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THE

Elements of the Ellipse,

TOGETHER WITH

THE RADII OF CURVATURE,

&c.

RELATING TO THAT CURVE;

AND OF

CENTRIPETAL AND CENTRIFUGAL FORCES

IN

ELLIPTICAL ORBITS;

TO WHICH IS ADDED,

THE FIRST OF DR. MATTHEW STEWART'S

Tracts.

By JAMES ADAMS,

LONDON:

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1818.

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THE

ELEMENTS OF THE ELLIPSE,

TOGETHER WITH

THE RADII OF CURVATURE,

&c.

PREFACE.

At a time when Geometry is at its zenith, there doubtless appears but little occasion for a publication on the Conic Sections; and I therefore beg to offer the following reasons as an apology for my now presenting a Treatise on this subject to the public.

Having devoted the little leisure afforded me by an active employment, to the consideration of the Nature and Properties of the Radii of Curvature, &c. relative to the Conic Sections, I found that the generality of authors had confined themselves to the demonstration of the first principles.

It appeared to me also, that the Demonstrations of Hamilton, T. Newton, Robertson, and others, were too abstruse for learners; and that if a simple Definition of the Circle of Curvature were substituted in their place, a series of Propositions might be collected and arranged so as to render the study of this portion of Geometry more pleasing and less laborious.

In making this collection and arrangement, however, I perceived that there would be occasion to refer to more than one treatise on the Conic Sections, and I consequently resolved to select from the before mentioned, and a few other authors, some of the most useful properties of the Ellipse.

As any property of the Curve may be assumed as a Definition, I have chosen that which is used by the Rev. T.

Newton in his excellent little treatise on the Conic Sections.

The Proposition in which the Ellipse is formed by a plane cutting a cone, is placed near the commencement, (it being the 4th), in order to compare it with that which immediately precedes it, and thereby establish the validity of the first Definition.

It here occurred to me, that some observations on the Centripetal and Centrifugal forces, in relation to Elliptical Orbits, might be usefully introduced, and I accordingly selected a few Propositions for their elucidation.

The utility of the first of Dr. Matthew Stewart's Tracts in reference to the forégoing subjects, together with its scarcity, has induced me to republish it at the end of this work.

A variety of useful Propositions may in a similar manner be deduced from the Hyperbola and Parabola.

Stonehouse,
near Plymouth.

TO

THOMAS BEWES, ESQ.

THIS

Treatise

IS INSCRIBED,

BY

HIS MUCH OBLIGED

AND

VERY HUMBLE SERVANT,

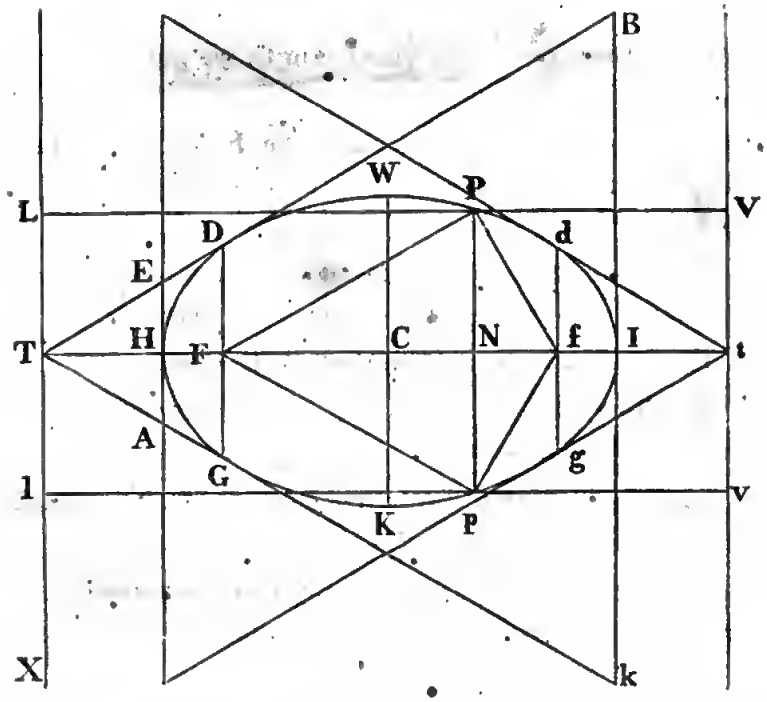
THE AUTHOR.

THE

ELEMENTS OF THE ELLIPSE.

OF THE ELLIPSE.

DEFINITIONS.



1. If any point F, be assumed without the line LX, and whilst the line FP revolves about F, as a center, a point P, moves in it in such a manner, that its distance from F, shall always be to PL (at right angles to LX) in a *given ratio*, the curve described by the point P, is called an *Ellipse*, FP being less than PL. See Proposition IV.

2. The line L X, is called the *Directrix.*

3. The point F, is called the *Focus.*

4. The line H I, drawn through the focus perpendicular to the directrix, is called the *Axis.*

5. The points H and I, where the curve meets the axis, are called the *Vertices.*

6. The line D F G, drawn through the focus parallel to the directrix, is called the *Principal Parameter,* or *Latus-rectum.*

7. The tangents T D B, T G k, which are drawn through the extremities of the latus-rectum, are called *Focal Tangents.*

8. The right line H I, is called the *Transverse Axis,* or *Axis Major.*

9. If the transverse axis be bisected in C, the point C, is called the *Center.*

10. If K W, which is bisected in C, be drawn perpendicular to H I, and $C W^2 = C K^2 = H F \times F I$; K W is called the *Conjugate Axis,* or *Axis Minor.*

11. A right line P N p, drawn parallel to the tangent A H E, or perpendicular to H I, is called an *Ordinate to the Axis.*

12. The segments I N, N H, are called *Abscissas.*

13. Any line passing through the center, and terminated by the curve, is called a *Diameter.*

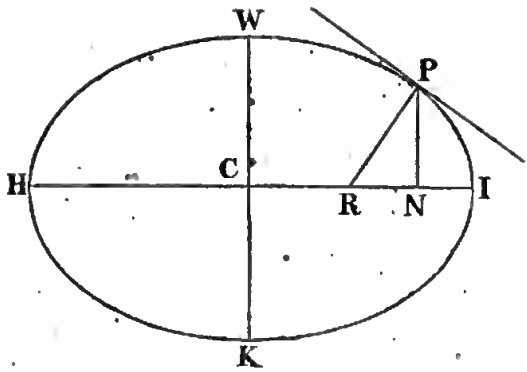
14. The points where the diameter meets the curve, are called the *Vertices* of that diameter.

15. If a right line be drawn from any point in a diameter parallel to the tangent at its vertex and terminated by the curve, it is called an *Ordinate* to that *Diameter.*

16. The segments of any diameter, between the ordinates and vertices, are called *Abscissas.*

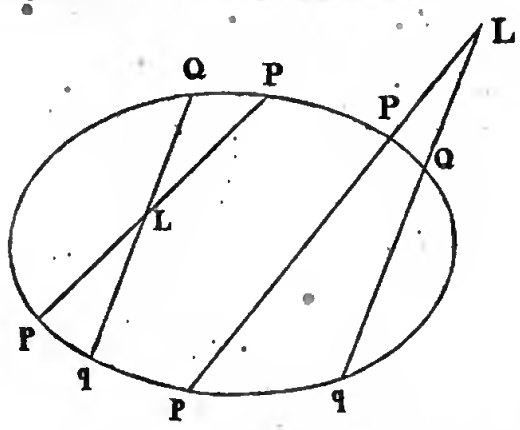
17. A diameter, which is parallel to the tangent at the vertex of any diameter, is called a *Conjugate Diameter.*

18. A line, which is a third proportional to any diameter and its conjugate, is called a *Parameter* of that *Diameter.*



19. The right line P R, drawn perpendicular to the tangent, and intersecting the axis in R, is called a *Normal.*

20. The segment N R, is called a *Subnormal.*



Def. 21. Right lines L P p, L Q q, each meeting the section in two points, are called *Secants;* and L, the point of intersection, is called the *Point of Concourse.*

Corollary 1. *(Fig. page 1.)*

When the revolving line F P, comes into the position F H T; F P, P L, will become F H, H T; therefore F H : H T :: F P : P L.

Cor. 2. When F P comes into the position of F D; F P, P L, will become F D, F T; therefore F D : F T :: F P : P L. Hence the latus-rectum D G is bisected in F.

Cor. 3. The Ellipse has two foci and two directrices. For if

in H I produced, I f be made equal to H F, and I t to H T, and V t v be drawn at right angles to H I, or parallel to L T l, and if f P revolve about f, in such manner, that f P, P V, have always the same *given ratio*, as F P, P L; a *curve* would be described by the point P, similar and equal to *that* described by P, by the revolution of the line F P about the focus F. Therefore (Def. 2 and 3) V t v is another directrix, and f another focus.

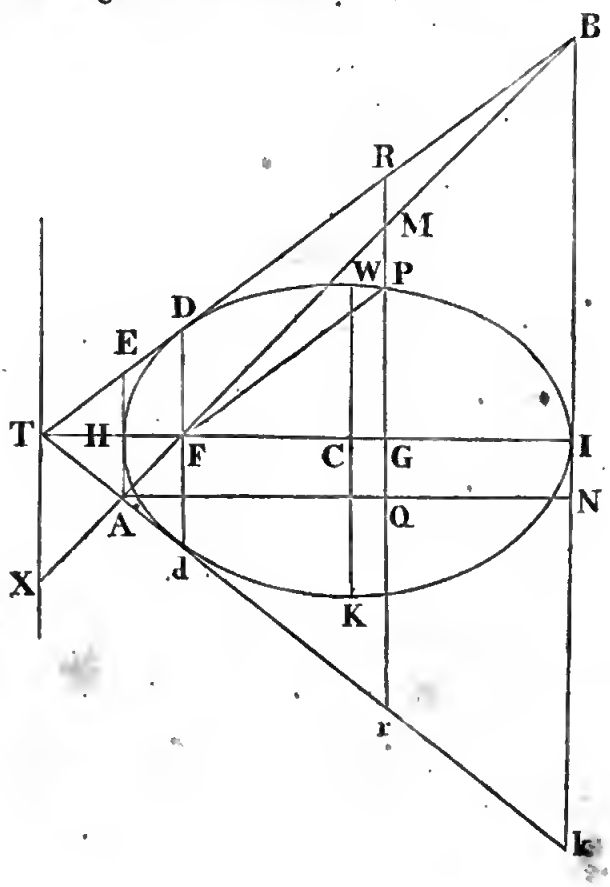
Cor. 4. Since H C=H I (def. 9) and F H=f I (cor. 3), it follows that C F=C f, and C T=C t.

Cor. 5. C F+C I=C f+C H=H f=I F.

Cor. 6. I F × H F=H f × f I=C K²=C W² (def. 10).

Proposition I.

If T E D, T A d be focal tangents and a right line be drawn from any point P to the focus F, and the perpendicular R G P, meeting the tangent in R. Then P F=R G.



For the triangles T G R, T F D are similar.

Ther. G R : G T :: F D : F T :: F P : G T (cor. 2. def.)

Hence P F = R G.

Cor. If H E be drawn parallel to the directrix, then H F = H E. This is plain from the proposition.

Prop. II.

If from A where EH meets the tangent Td, a straight line be drawn through A and F, it will meet the tangent T D somewhere in B.

Produce F A to X. Then, because the triangles F H A, F T X are similar, and F H = H A (cor. prop. 1.) F T = T X; but F D is less than F T (def. 1.) Ther. F D is less than T X, consequently the lines T D, X F must meet when produced.

Cor. I. If A F meet R G in M, then F G = G M. Because the triangles F G M, F H A are similar, and F H = H A; F G = G M.

Cor. 2. When R coincides with B, then R G = B I = F I; but F P will become equal to F I (Prop. I.) Therefore the curve will meet the axis in I.

Prop. III.

The rectangle under the abscissæ, is to the square of the semi-ordinate to the axis, as the square of the transverse axis is to the square of the conjugate axis.

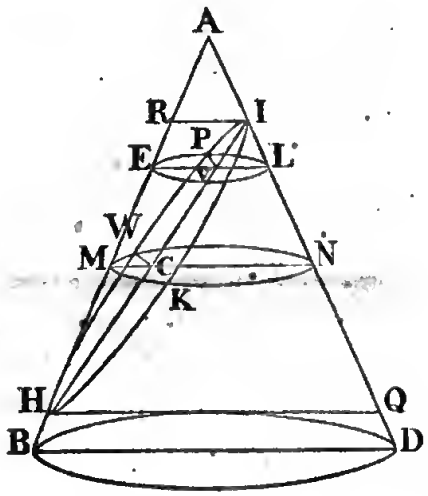
Draw A N parallel to H I; then on account of the parallels, E H A, R G r, B I k, R M : E A :: B M : B A :: Q N : A N :: G I : H I; and M r : B k :: A M : A B :: A Q : A N :: H G : H I. Therefore R M × M r : E A × B k :: G I × H G : H I^2; but (5. 2. e) R M × M r + M G^2 = R G^2 = F P^2 and M G^2 = F G^2 (cor. I. prop. 2.) ther. R M × M r + F G^2 = F P^2; ther. R M × M r = P G^2. A E = 2 H F; B k = 2 F I; and (per def. 10.) H F × F I = C W^2; hence by substitution P G^2 : 4 C W^2 :: H G × G I : H I^2; that is, H G × G I : P G^2 :: H I^2 : K W^2.

Cor. 1. Because H I and K W, are bisected in C; H G × G I : P G^2 :: C I^2 : C K^2.

Cor. 2. The axis being constant; PG^2 varies as $HG \times GI$.

Cor. 3. As the property is the same for the ordinates on both sides of the axis, at equal distances from the center, or from the vertices, the ordinates on both sides are equal; or the double ordinates are bisected by the axis.

Prop. IV.



If a cone ABD be cut through its two sides by a plane HWPIK, the section will be an Ellipse.

Bisect HI in C, and let a section MWNK be drawn through C parallel to the base, and also another section EPLV parallel to MWNK; then will these sections be circles; hence, $EV \times VL = VP^2$, and $MC \times CN = CW^2$. Draw HQ, RI parallel to MN or EL;

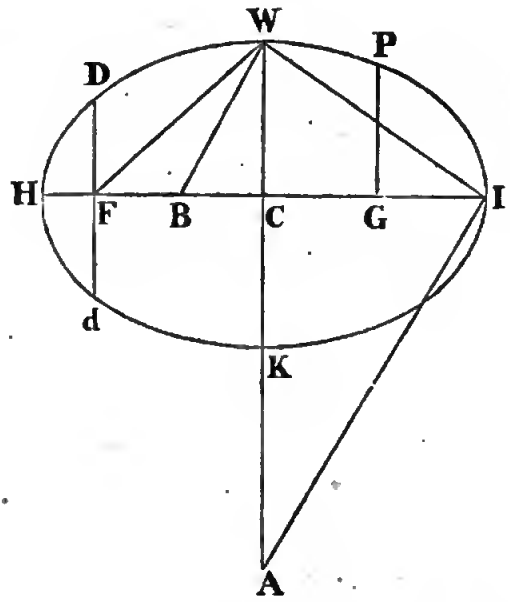
Then by similar triangles, HC : CM :: HV : VE; and IC : CN :: IV : VL.

Therefore $HC \times CI : CM \times CN :: HV \times VI : VE \times VL$.

That is $CI^2 : CW^2 :: HV \times VI : VP^2$; which is the property of the Ellipse by Prop. 3. Hence, according to Def. 1, the *Ellipse* is a conic section.

Prop. V.

The latus-rectum is a third proportional to the transverse and conjugate axes.



For $HC^2 : CK^2 :: HF \times FI : FD^2$ (Cor. 1. Prop. III.)

But $HF \times FI = CK^2$ (def. 10.) ther. $HC^2 : CK^2 :: CK^2 : FD^2$,

Or $HC : CK :: CK : FD$.

Cor. 1. $HG \times GI : GP^2 :: HI :$ latus-rectum.

For $HI : KW :: KW : Dd$; or $HI : KW^2 :: 1 : Dd$; ther. $HI^2 : KW^2 :: HI : Dd :: HG \times GI : PG^2$.

Cor. 2. If IW be joined, and WB and IA, be drawn at right angles thereto, intersecting the axes in B and A, then CB will represent the semi-parameter of the transverse axis, and CA, the semi-parameter of the conjugate axis. For from the nature of the right-angled triangles BIW and AIW; $CB = \frac{CW^2}{CI}$, and $CA = \frac{CI^2}{CW}$.

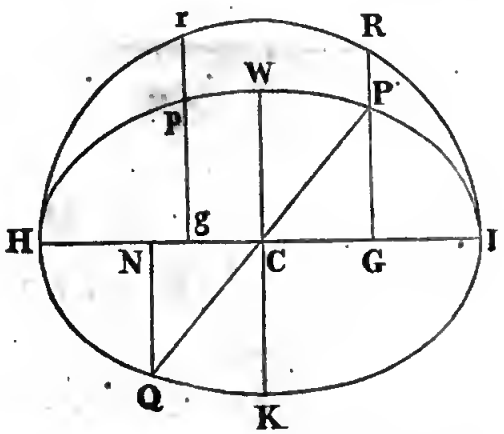
Prop. VI.

A line drawn from either of the foci to the extremity of the conjugate axis, is equal to the semi-transverse axis.

By Cor. 1. Prop. III. $CH^2 : CW^2 :: HC^2 - CF^2 : FD^2$, and by Prop. V. $CH^2 : CW^2 :: CW^2 : FD^2$. Ther. $HC^2 - CF^2 : FD^2 :: CW^2 : FD^2$; ther. $HC^2 - CF^2 = CW^2$; $FW^2 - CF^2 = CW^2$, Ther. $HC^2 - CF^2 = FW^2 - CF^2$; ther. $HC = FW$.

Prop. VII.

If a circle H R I be described on the transverse axis H I, and G P an ordinate be produced to R, it will be as G R : G P :: C I : C K.



By a property of the circle $HG \times GI = GR^2$. And by Cor. 1. Prop. III. $HG \times GI : GP^2 :: CI^2 : CK^2$; ther. $GR^2 : GP^2 :: CI^2 : CK^2$, that is, $GR : GP :: CI : CK$.

Cor. Draw any other ordinate g p, and produce it to r. Then because C I and C K are constant, G R : G P :: g r : g p.

Prop. VIII.

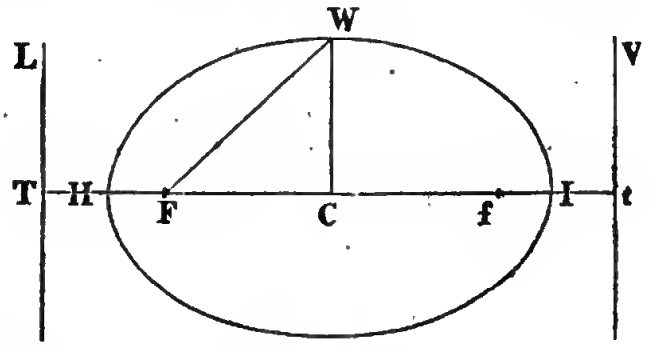
All the diameters are bisected in the center.

Draw P C to the center C, and make C N = C G, also draw N Q parallel to P G, which produce to meet the curve in Q, and

join C Q. Then because C N=C G; Q N=P G (Cor. 3. Prop. III.) and the angles at G and N are right angles, the triangles C N Q, C G P are equal. Ther. C Q=C P; and the angles N C Q, G C P being equal, P C Q is a straight line bisected in C.

Prop. IX.

If H I be the transverse, and C W the semi-conjugate axis, C the center, F f the foci, and L T, V t the directrices; then H I : T t :: H F : H T.



For F H : H T :: F I : I T (def. 1.)

Ther. F H + F I : H T + I T :: H F : H T.

That is, H I : T t :: H F : H T.

Cor. 1. Since T t and H I are bisected in C; C H : C T :: H F : H T.

Cor. 2. C F : C H :: H F : H T.

For F I : I T :: F H : H T :: f I : H T; ther. F I − f I : I T − H T :: f I : H T.

That is, F f : H I :: f I : H T; or C F : C H :: H F : H T.

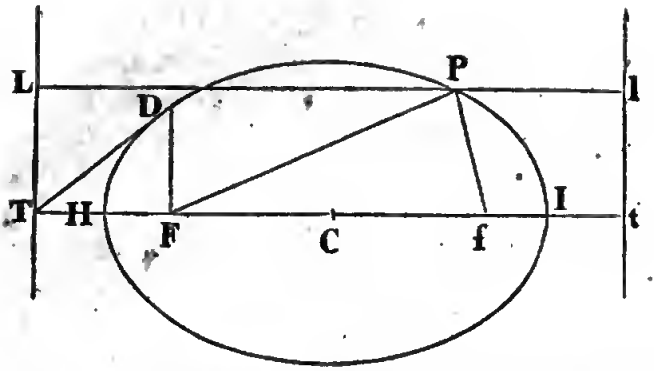
Cor. 3. C F : C H :: C H : C T.

For H I : T t :: H F : H T; or C H : C T :: H F : H T; ther. by cor. 2. C F : C H :: C H : C T.

Cor. 4. Since C F × (C F + F T) = C H² = C F² + C W² (Prop. VI.) or, C F² + C F × F T = C F² + C W²; it follows C F × F T = C W².

Prop. X.

If from any points in the curve, two right lines be drawn to the foci, their sum is equal to the transverse axis.



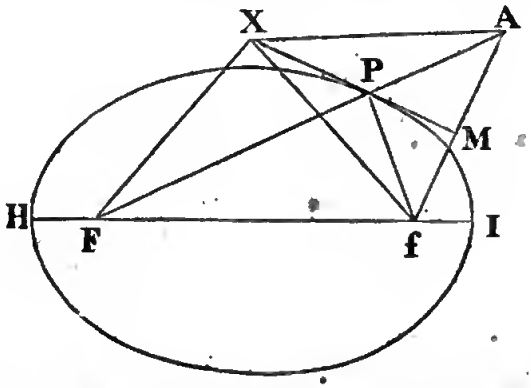
Let H I be the transverse axis, and F, f, the foci, and T L, t l, the directrices; from any point P, draw P F, P f, and through P, draw L P l parallel to H I; then will L l be equal to T t.

By def. 1. and cor. 2 and 3. prop. IX. P F : P L :: C F : C H : C H : C T :: 2 C H : 2 C T :: H I : T t. But as P F : P f :: P L : P l; we have P F + P f : P F :: P L + P l : P L; that is, P F + P f : T t :: P F : P L;

Ther. H I : T t :: P F + P f : T t; consequently P F + P f = H I.

Prop. XI.

If from any point P in the curve, P F, P f be drawn to the foci; the line P M, bisecting the angle f P A, will touch the curve in the point P.



Produce F P to A, so that P A equal P f, and let a point X be taken in the line P M, and draw X F, X f, X A, and also f A meeting P M in M. Then because in the triangles P M f, P M A; P f = P A and P M common to both, and the angles M P A, M P f

equal; M f=M A, and the angles at M right angles; therefore X f=X A; but X F + X A are greater than F A, that is, than F P + P F or H I (Prop. X.), therefore the point X is not in the curve, and consequently the line P M touches the curve in the point P.

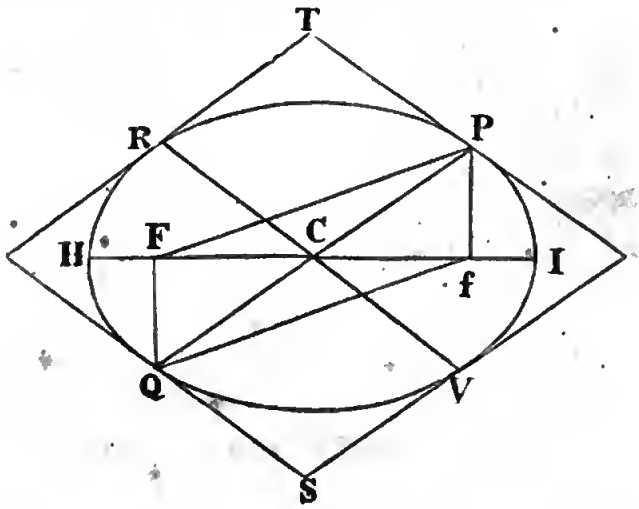
Cor. 1. If from any point P in the curve two right lines be drawn to the foci, they will make equal angles with the curve, or with the tangent drawn through the point P.

For the ∠ A P M=F P X, ther. the ∠ f P M=F P X.

Cor. 2. If from the focus f, f M be drawn at right angles to the tangent P M, and intersect F P produced in A, then F A will be equal to the transverse axis H I. For P A = P f, and (Prop. X.) F P + P f=F P + P A = F A=H I.

Prop. XII.

The tangents at the vertices of any diameter are parallel.



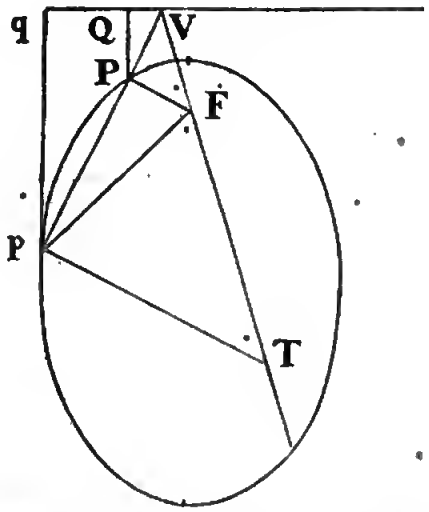
Let P C Q be any diameter, H I the transverse axis, F, f, the foci, and C the center. Draw the tangents P T, Q S; F P, P f, f Q and Q F. Then because F C=C f (cor. 4. def.) and C P= C Q (Prop. VIII.) also the ∠ P C f=F C Q, and ∠ F C P=f C Q, it follows that F P is equal and parallel to Q f, and F Q equal and parallel to P f; ther. F P f Q is a parallelogram, hence the

$\angle$ F P f = F Q f. and the halves of their supplements will be equal (Cor. Prop. XI.) that is, the $\angle$ F P T = f Q S, to which add the equal angles F P Q, f Q P, and the $\angle$ Q P T = P Q S; therefore P T is parallel to Q S.

Cor. If tangents be drawn at R and V, the vertices of any other diameter R C V, and produced to meet the tangents P T and Q S, a parallelogram will be formed circumscribing the Ellipse.

Prop. XIII.

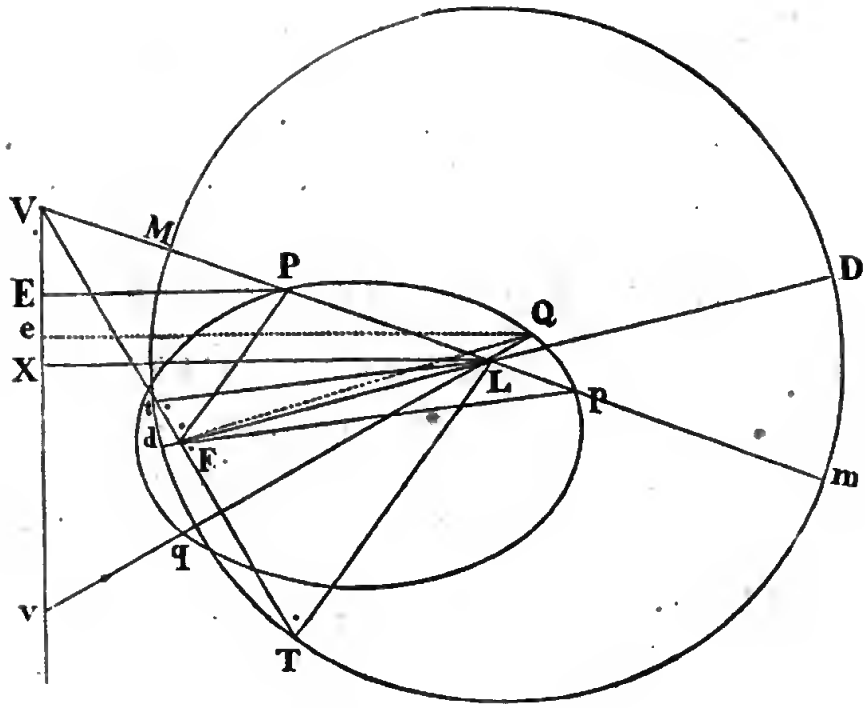
If a right line P p, which cuts the curve in P, p, meet the directrix in V, and a right line V F T be drawn through the focus, and F P, F p joined; then the $\angle$ P F V = $\angle$ p F T.



Draw p T parallel to P F to meet v F in T; and draw P Q, p q perpendicular to the directrix. Then the triangles V P Q, V p q are similar; as are also the triangles V F P, V T p. And F P : P Q :: F p : p q (def. 1.) Or F P : F p :: P Q : p q :: P V : p V :: P F : p T, ther. p T = p F; and the angle p F T = p T F, which is equal to P F V (by construction).

Prop. XIV.

If two right lines P p, Q q intersect each other in L, and meet the directrix in V, v; then P L × L p and Q L × L q will be in a constant ratio to each other, wherever L, the point of concourse be taken.



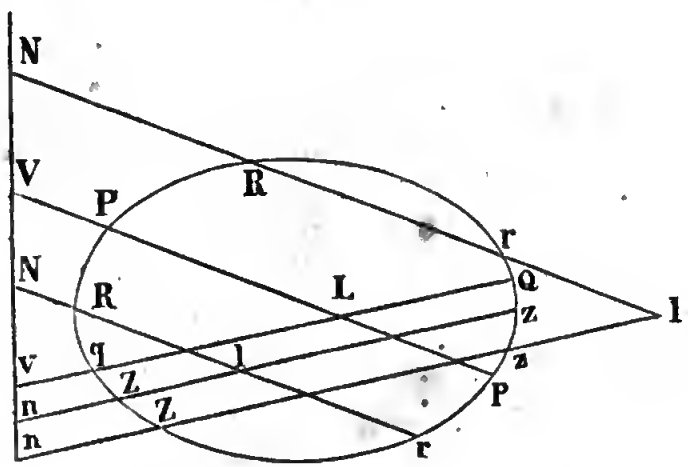
Let F be the nearest focus to the directrix V v; join V F and produce it, also join F P, F p; draw P E, L X perpendicular to the directrix; draw L T parallel to P F, and L t parallel p F meeting V F in T and t.

$\angle$ P F V = p F T (Prop. XIII.) ther. $\angle$ L T t = L t T; hence L T = L t. From the center L, and distance L T or L t, describe a circle, cutting V P p, in M and m; join F L, and produce it to meet the circle in D and d. The triangles V P E and V L X are similar, as are also the triangles V P F, V L T. Ther. L T : P F :: L V : P V :: L X : P E; that is, L T : L X :: F P : P E, a given ratio (Prop. I.) Ther. $LT = \frac{FP}{PE} \times LX$; hence L T varies as L X. Because L T is parallel to P F, and L t to p F; L P : T F :: L V : T V, and p L : F t :: L V : t V. Ther. L P $\times$ L p : F T $\times$ F t :: L V^2 : V T $\times$ V t. But F T $\times$ F t = F D $\times$ F d; and V T $\times$ V t = V m $\times$ V M = V L^2 $-$ M L^2, ther. by substitution, L P $\times$ L p : F D $\times$ F d :: V L^2 : V L^2 $-$ M L^2. But L V :

LT (LM) :: PV : PF, and $LV^2 : LM^2 :: PV^2 : PF^2$; ther. by division $LV^2 : LV^2 - LM^2 :: PV^2 : PV^2 - PF^2 :: LP \times Lp : FD \times Fd$. Per trig. PV : PE :: Rad. : Sin. V. whence $PV^2 = \frac{PE^2}{\text{Sin.}^2 \angle V}$ (radius 1.) $= \frac{PE^2}{s^2}$, ther. by sub. for PV^2, we have $PL \times Lp : DF \times Fd :: \frac{PE^2}{s^2} : \frac{PE^2}{s^2} - PF^2 :: PE^2 : PE^2 - s^2 \times PF^2 :: 1 : 1 - \frac{PF^2}{PE^2} \times s^2 :: 1 : 1 - v^2 s^2$. That is, $PL \times Lp : FD \times Fd :: 1 : 1 - v^2 s^2$ ($v^2 = \frac{PF^2}{PE^2}$; and $s^2 = \text{sin.}^2 LVX$.) In the same manner it may be proved that, $QL \times Lq : FD \times Fd :: 1 : 1 - \frac{QF^2}{QE^2} \times \text{sin.}^2 LvX$. That is, $QL \times Lq : FD \times Fd :: 1 : 1 - v^2 r^2$; (by joining QF, drawing Qe perpendicular to VX, and putting $r^2 = \text{sin.}^2 LvX$.)

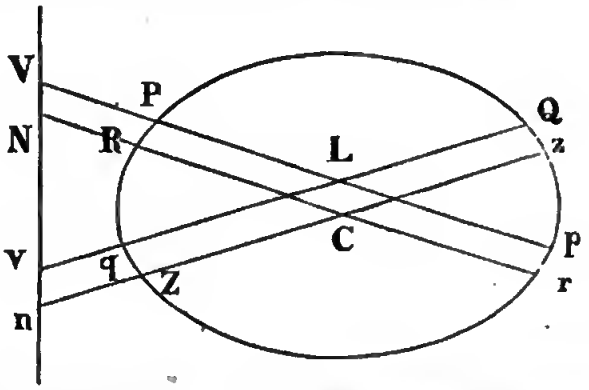
Hence $PL \times Lp : QL \times Lq :: 1 - v^2 r^2 : 1 - v^2 s^2$.

Cor. 1. If any other two right lines Rr, Zz, intersect each other in l, (either within or without the curve), being respectively parallel to Pp, Qq, their rectangles will be in the same given ratio, that is, $PL \times Lp : QL \times Lq :: Rl \times lr : Zl \times lz$.

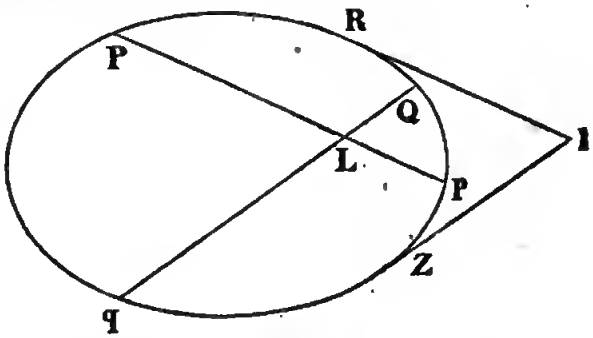


For the parallel lines make the angles at N and n with the directrix respectively equal to the angles at V and v.

Cor. 2. If the right lines R r, Z z, which are respectively parallel to P p, Q q; intersect each other in the center C. Then P L × L p : Q L × L q :: C R² : C Z². Because R r, Z z; make the same angles with the directrix, as P p, Q q, and are each bisected in the center (Prop. VIII.) the corollary is manifest.

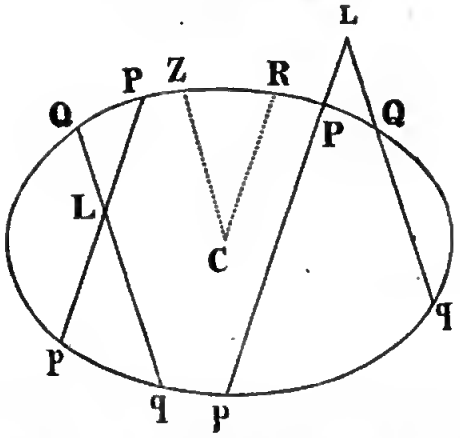


Cor. 3. If the lines P p, Q q, move parallel to themselves until they touch the curve in R and Z, and meet in l; then P L × L p : Q L × L q :: R l² : Z l². For when P p touches the curve, P, and p will coincide in R, and P L = R l = p l; ther. P L × L p = R l². In like manner, Q L × L q = Z l².

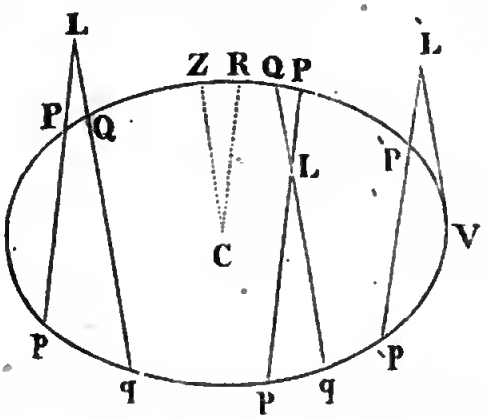


Cor. 4. If two right lines meeting each other, be respectively parallel to two diameters, according to their both touching, or both cutting, or one of them touching and the other cutting the curve, the squares of the segments of the tangents, or the rectangles under the segments of the secants, will be in a constant ratio to each other, wherever the point of concourse be taken.

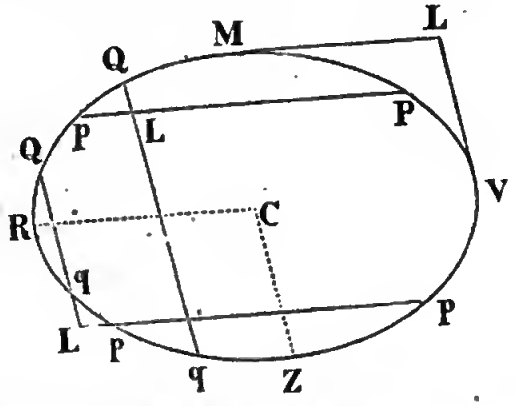
SCHOLIUM.



If four secants, intersecting each other, are respectively parallel to two semi-diameters; then $PL \times Lp : QL \times Lq :: CR^2 : CZ^2$

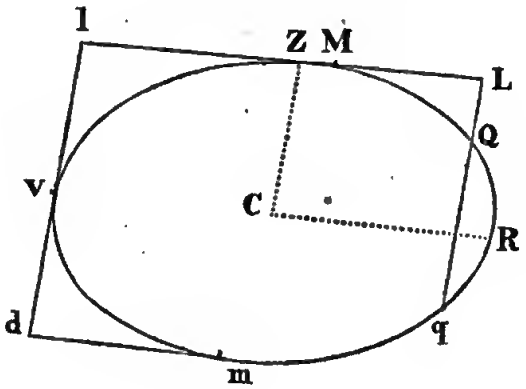


If three of the lines be secants, and one a tangent, then $PL \times Lp : QL \times Lq :: \text{`}L\text{`}P \times \text{`}L\text{`}p : \text{`}LV^2 :: CR^2 : CZ^2$.

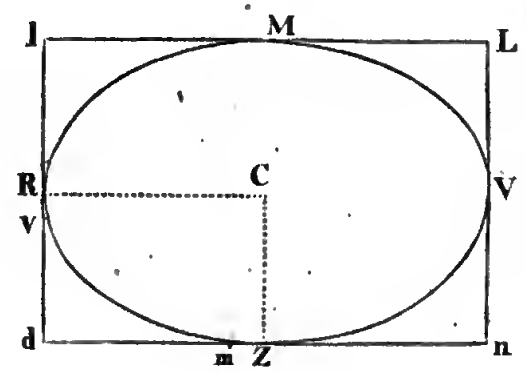


If two of the lines be secants and two tangents, then $PL \times Lp : QL \times Lq :: LM^2 : LV^2 :: CR^2 : CZ^2$.

If one of the lines is a secant and three tangents, then $dm^2 : dv^2 :: LM^2 : LQ \times Lq :: CR^2 : CZ^2$; $lM^2 : lv^2 :: LM^2 : LQ \times Lq :: CR^2 : CZ^2$. Ther. $dm : dv :: lM : lv$.



If the four lines be tangents, and touch the curve in M, V, m, v; then, $dm : dv :: lM : lv :: LM : LV :: nm : nV :: CR : CZ$.



LEMMA I.

If the straight line LM be so divided in Q and q that $LQ \times Lq = Mq \times MQ$; or $LQ \times QM = Mq \times qL$; then $LQ = Mq$.

First, bisect LM in N.

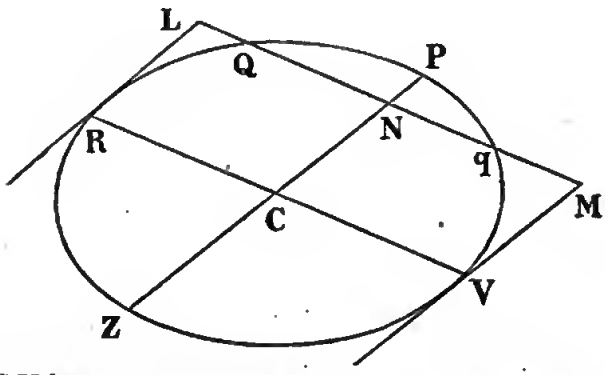
Then (5. 2. E.) $LQ \times QM + QN^2 = LN^2 = Mq \times qL + qN^2$.

But $LQ \times QM = Mq \times qL$; ther. $QN^2 = qN^2$; that is, $QN = Nq$. Ther. $LN - QN = MN - qN = LQ = Mq$.

Secondly, bisect Qq in N; then (6. 2. E.) $Lq \times LQ + QN^2 = LN^2$, and $MQ \times Mq + qN^2 = MN^2$; but $LQ \times Lq = Mq \times MQ$, and $QN^2 = Nq^2$; ther. $LN^2 = NM^2$; that is, $LN = NM$. Hence $LN - QN = MN - qN = LQ = Mq$.

Prop. XV.

All right lines drawn parallel to any diameter, and which are terminated by the curve, are bisected by the conjugate diameter.



Let RCV be any diameter, and PCZ its conjugate; through the vertices R, V, draw the tangents RL, VM; and through any point N in PZ, draw LNM parallel to RV, meeting the curve in Q and q, and the tangents in L and M. On account of the parallel lines, $RL = VM$, and $LN = NM$; and (Cor. 4. Prop. XIV.) $RL^2 : MV^2 :: LQ \times Lq : MQ \times Mq$; ther. $LQ \times Lq = MQ \times Mq$; hence by the Lemma, $LQ = Mq$; ther. $LN - LQ = MN - Mq = QN = Nq$.

Prop. XVI.

If PCZ be any diameter, RCV its conjugate, and QN a semi-ordinate. Then $PN \times NZ : QN^2 :: CZ^2 : CV^2$.

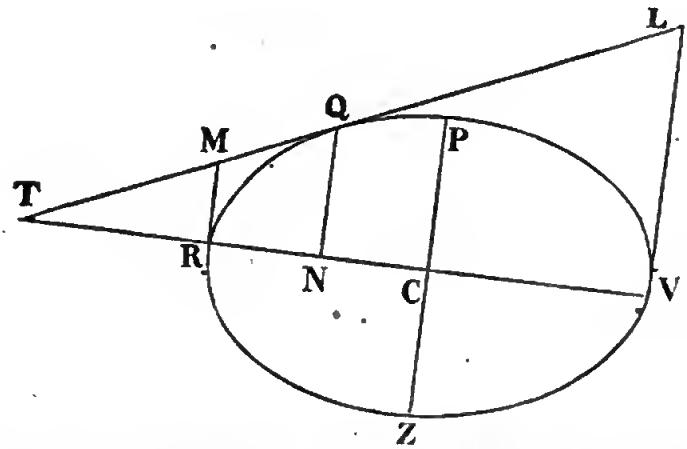
For (by Cor. 4. Prop. XIV.) $PN \times NZ : QN \times Nq\ (= QN^2) :: CZ^2 : CV^2$.

Cor. 1. Since $PN \times NZ : QN^2 :: CZ^2 : CV^2 :: CZ : \frac{CV^2}{CZ} :: 2CZ : \frac{2CV^2}{CZ}$; ther. $PN \times NZ : QN^2 :: PZ :$ Param. of PZ (def. 18.).

Cor. 2. C Z and C V being constant, P N × N Z will vary as Q N².

Prop. XVII.

If Q T be a tangent at any point Q, and meet the diameter V R in T; and Q N be a semi-ordinate, then C N : C R :: C R : C T.



Draw the tangents RM, VL to meet the tangent Q T in M and L.

Then (Cor. 4. Prop. XIV. and the parallel lines R M, N Q and V L) M Q : Q L :: M R : L V :: R T : V T :: R N : N V; by division, V T − R T : R T :: N V − R N : R N; that is, R V : R T :: N C + C R − R N : R N.

Or, R V : R T :: 2 N C : R N; or, C R : R T :: N C : R N; by composition, C R + R T : C R :: C N + N R : C N.

That is, C T : C R :: C R : C N; or, C N : C R :: C R : C T.

Cor. 1. R N : N C :: T N : N V.

For C N : C V :: C V : C T; by composition, C N + C V : C N :: C V + C T : C V.

That is, N V : N C :: V T : V C; or, C N : C R : N V : V T; by division, C R − C N : C N :: V T − V N : V N;

That is, R N : N C :: T N : N V.

Cor. 2. T R : T N :: T C : T V.

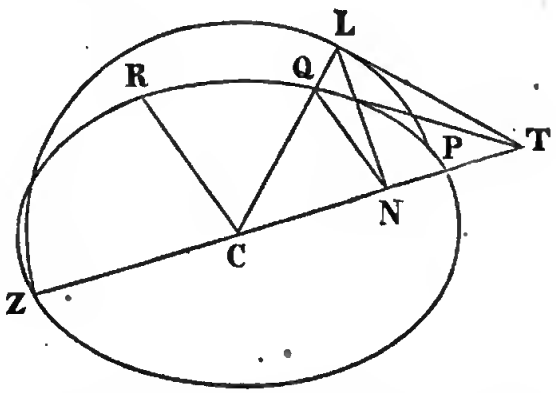
For C T : C R :: C R : C N; by division, C T − C R : C T :: C R − C N : C R.

That is, T R : C T :: R N : C R; by composition, T R + R N : T R :: C T + C R : C T.

Or, T N : T R :: V T : T C; T R : T N :: T C : T V.

Prop. XVIII.

If a circle Z L P be described on the diameter P Z, and N L be drawn perpendicular thereto, and N Q an ordinate to the curve; then, N L : N Q :: C Z : C R.

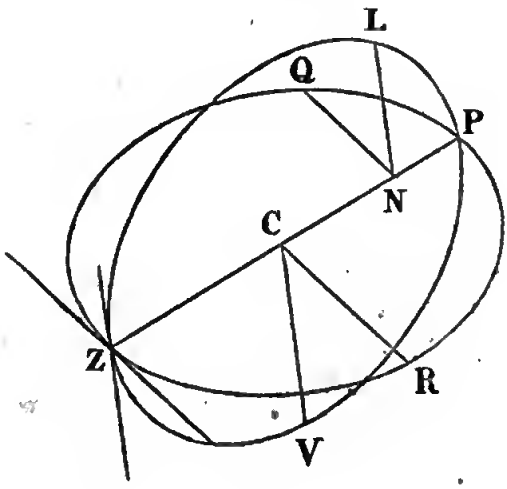


For, by Prop. XVI. P N × N Z : N Q² :: C Z² : C R². And, by a property of the circle, P N × N Z = N L²; ther. N L² : N Q² :: C Z² : C R²; that is, N L : N Q :: C Z : C R.

Cor. C Z and C R being constant, C L will vary as N Q.

Prop. XIX.

If two Ellipses Z L P V, Z Q P R, be described on the same diameter P Z, and the ordinates N L, N Q be drawn, and the semi-diameters C V, C R parallel to these ordinates be also drawn, then N L : N Q :: C V : C R.



By Prop. XVI. $PN \times NZ : NL^2 :: CP^2 : CV^2$.

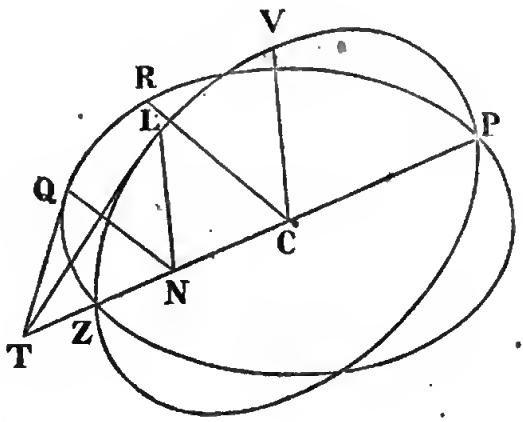
And, $PN \times NZ : NQ^2 :: CP^2 : CR^2$.

Therefore, $NL^2 : CV^2 :: NQ^2 : CR^2$.

That is, $NL : NQ :: CV : CR$.

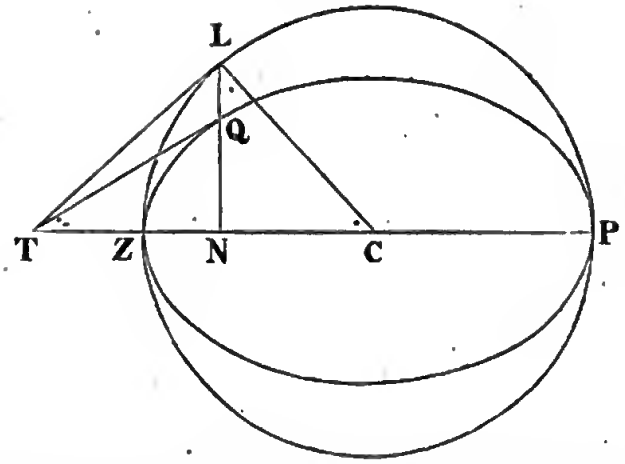
Prop. XX.

If ZLVP, ZQRP be two Ellipses, having a common diameter, PZ; NL, NQ ordinates thereto, and CV, CR semi-diameters parallel to these ordinates; tangents drawn at L and Q will intersect each other in T, a point in the diameter PZ, produced.



Since PZ is a common diameter to both curves, and the ordinates are drawn from the same point N, and by Prop. XVII. $CN : CZ :: CZ : CT$; then TL, TQ are tangents at the points L and Q.

Cor. 1. If a circle ZLP, and an ellipse ZQP have a common diameter PZ and a common abscissa ZN; tangents at the extremities of the ordinates NL, NQ will meet each other in T, a point in the diameter PZ, produced. Draw LT a tangent to the circle and join TQ, and CL; then because the triangles CLT and CNL are similar, $CN : CL :: CL : CT$. Or, $CN : CZ :: CZ : CT$. Therefore TQ touches the ellipse in Q.

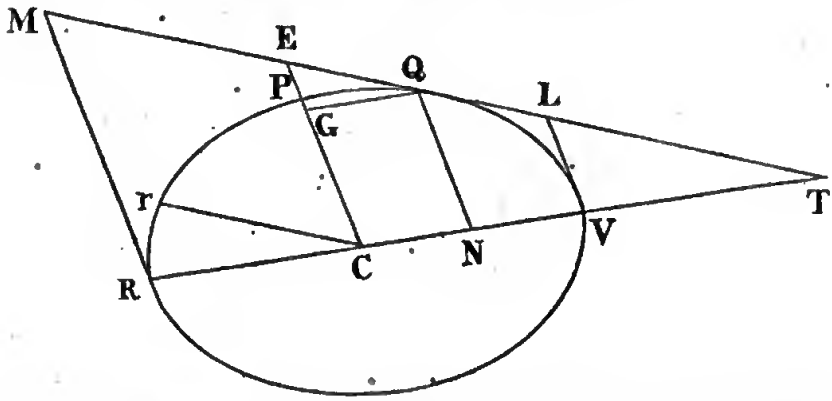


Cor. 2. If one of the curves be a circle and the other an ellipse having the ordinates N L, N Q as in fig. to Prop. XVIII. tangents at L and Q, will meet each other in T, for the same reason as in Cor. 1.

Prop. XXI.

Draw through the extremities of any diameter R V, two tangents to meet any other tangent Q M, in the points L, M; and let C P be conjugate to R V and C r parallel to L M.

Then $RM \times VL = CP^2$; and $MQ \times QL = Cr^2$.



Case 1. Draw the ordinates Q N, Q G; by Prop. XII. R M and V L are parallel, and by Prop. XVII. $CN \times CT = CV^2$ Therefore $CT^2 - CN \times CT = (CT - CN) \times CT = NT \times CT$. And $CT^2 - CV^2 = RT \times VT$; ther. $RT \times VT = NT \times CT$; Hence R T : T N :: C T : V T :: C E : : V L (sim. triangles). But R T : N T : : R M : N Q. Ther. R M : N Q : : C E : V L, ther. $RM \times VL = CE \times NQ = CE \times CG = CP^2$ (Prop. XVII.)

Case 2. Because (Cor. 4. Prop. XIV.) $MR : MQ :: LV : LQ$; it will be $MR^2 : MR \times LV :: MQ^2 : MQ \times LQ$; by substitution, $MR^2 : CP^2 :: MQ^2 : MQ \times LQ$; but (Cor. 4. Prop. XIV.) $MR^2 : MQ^2 :: CP^2 : Cr^2$; ther. $CP^2 : Cr^2 :: CP^2 : MQ \times LQ$; ther. $MQ \times LQ = Cr^2$.

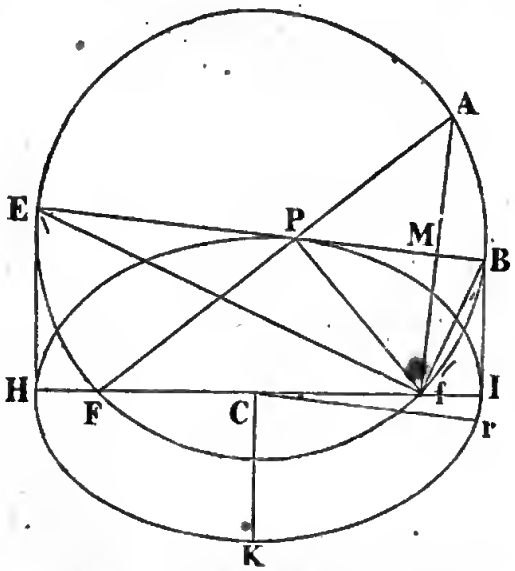
Prop. XXII.

If a tangent at any point Q, meet the transverse and conjugate diameters CV, CP, in T and E, and Cr be drawn parallel to TM, then $EQ \times QT = Cr^2$.

Draw the ordinate QN, and through the vertices of the diameter RV, draw the tangents RM, VL, which (Prop. XII.) are parallel to each other, as well as to the ordinate NQ; by Cor. 1. Prop. XVII. $CN \times NT = RN \times NV$. Ther. $CN : NV :: RN : NT :: EQ : QL :: MQ : QT$ (sim. triang.) Hence $EQ \times QT = LQ \times QM = Cr^2$, (Prop. XXI.)

Prop. XXIII.

Draw through the extremities of the transverse axis HI, two tangents to meet any other tangent PM in the points E, B; then a circle described about EB as a diameter will pass through the foci F, f.



Let CK be the semi-conjugate axis, and join fB, fE, fP and FP, which last produce to A, and join fA which will intersect EB in M. Then (Def. 10. and Prop. XXI.) $Hf \times fI = HE \times$

IB=CK². Ther. HE : Hf :: If : IB. These lines are about the equal angles at H and I; therefore the triangles HEf, IBf are equiangular, and the angles BfI, fEH equal; but BfI+IBf=BfI+EfH=a right angle, therefore BfE is a right angle, and consequently the point f is in the circumference of the circle described about BE. In like manner it may be shewn that the other focus F is in the circumference of the same circle.

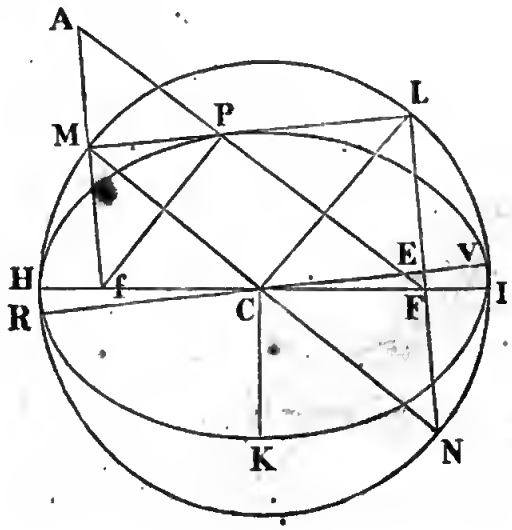
Prop. XXIV.

If PF, Pf, be drawn from any point P in the curve to the foci, and Cr parallel to the tangent EPB, then PF×Pf=Cr².

Draw the tangents HE, IB; then (Prop. XXIII.) a circle described about the diameter EB will pass through the foci. Draw from either focus as f, the perpendicular fM to EB, and let it meet FP in A; and because the angles fPM, APM are equal (Prop. XI.) and PM common; fM=MA; and Pf=PA; therefore because the diameter EB of the circle bisects fA, and is perpendicular to it, and the point f is in the circumference, the point A will be in the same; therefore by a property of the circle FP×PA=FP×Pf=EP×PB=Cr² (Prop. XXI.)

Prop. XXV.

If from the foci F, f; FL, fM be drawn perpendicular to the tangent PM, a circle described about the transverse axis, will pass through the points L, M, where the perpendiculars meet the tangent.



Let H I be the transverse axis and C the center and join f P, F P; let the perpendicular f M meet F P in A, and join C M. Because (Prop. XI.) the angles f P M, A P M are equal and P M common, f P=P A and thence, F A=H I (Prop. X.) But f M =M A; and likewise f C=C F (Cor. 4. Def.) ther. F A and C M are parallel, and thence C M=half F A=half H I. Therefore M is in the circumference of the circle described about H I; and in like manner the point L will be in the same circumference.

Cor. If the diameter R C V be parallel to the tangent P M, it will cut off from P F the segment P E, equal to half the transverse axis.

For C M P E is a parallelogram, ther. P E=C M=half H I.

Prop. XXVI.

If the perpendiculars F L, f M he drawn from the foci F, f, to the tangent P M, then will $F L \times f M = C K^2$.

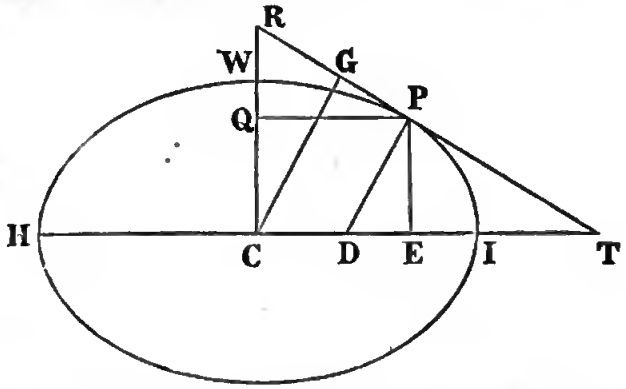
Let H I be the transverse and C K the semiconjugate axis, and C the center. Join C M, C L, and let C M meet F L in N. Because F N, f M are parallel, the triangles C f M, C F N are equiangular, and because C F=C f (Cor. 4. Def.); C M=C N, and F N=f M, C M=C L=C N; the point N, is therefore in the circumference of the circle described about the transverse axis H I as a diameter. Hence by a property of the circle $F L \times F N = F L \times f M = H F \times F I = C K^2$ (Def. 10.)

Cor. 1. $F L^2 = \frac{F P}{f P} \times C K^2$. For the triangles F P L, f P M are similar, ther. F P : F L :: f P : f M, or $F P \times f M = F L \times f P$, or $F P \times f M \times F L = F L^2 \times f P$. ther. $F L^2 = \frac{F P}{f P} \times f M \times F L = \frac{F P}{f P} \times C K^2$.

Cor. 2. $C K^2$ being constant; $F L^2$ will vary as $\frac{F P}{f P}$.

Prop. XXVII.

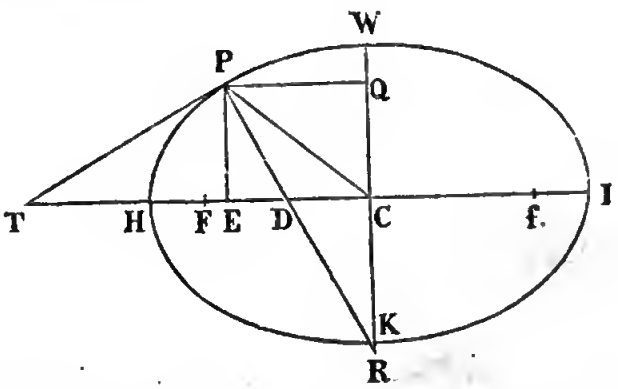
Let H I be the transverse and C W the semi-conjugate axis, C the center, and P T a tangent at P; let P D and C G be perpendiculars to the tangent. Then $Pd \times CG = CW^2$.



Draw P E, P Q perpendicular to the axes, and let C W meet P T in R; then Q C = P E; and on account of the parallel lines, the right-angled triangles C G R and P E D are similar, and C G : C R :: P E : P D, ther. $CG \times PD = CR \times PE = CR \times CQ = CW^2$ (Prop. XVII.)

Prop. XXVIII.

If P E be an ordinate to the transverse axis, and P D perpendicular to the tangent at P, and F, f, the foci; then H I : latus-rectum :: C E : E D.



For C E : E D :: C E × E T : E D × E T; but $CE \times ET = IE \times EH$ (Cor. 1. Prop. XVII.) And because D P T is a right angle, $ED \times ET = EP^2$, therefore C E : E D :: I E × E H : EP^2 :: H I : *latus-rectum*. (Cor. 1. Prop. V.)

Cor. 1. $CE : ED :: CH^2 : CW^2$. For $CE : ED :: HI : \frac{KW^2}{HI} :: HI^2 : KW^2 :: CH^2 : CW^2$.

Cor. 2. $CE : CD :: CH^2 : CF^2$. For $CE - ED : CE :: CH^2 - CW^2 : CH^2 :: CF^2 : CH^2$; that is, $CE : CD :: CH^2 : CF^2$.

Cor. 3. Produce PD to cut the conjugate axis WK in R, draw the ordinate PQ and join CP. Then $QR : QC :: CH^2 : CW^2$.

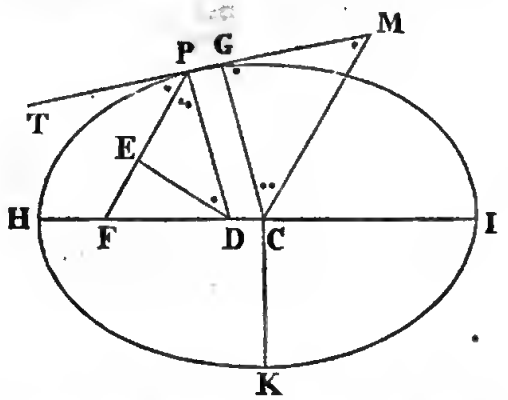
For by sim. triangles $QR : QP :: PE : ED$; or $QR : QC :: CE : ED :: CH^2 : CW^2$ (Cor. 1.)

Cor. 4. Tang. EDP : Tang. ECP :: $CI^2 : CK^2$.

For EC : ED :: Tang. EPC : Tang. EPD :: Cotang. EPD : Cotang. EPC :: Tang. EDP : Tang. ECP. That is, EC : ED :: Tang. EDP : Tang. ECP :: $CI^2 : CK^2$.

Prop. XXIX.

If a right line DE be drawn from the point D, where the normal meets the transverse axis, perpendicular to the distance PF, between the point of contact and focus, it will cut off the segment PE equal to half the latus-rectum.

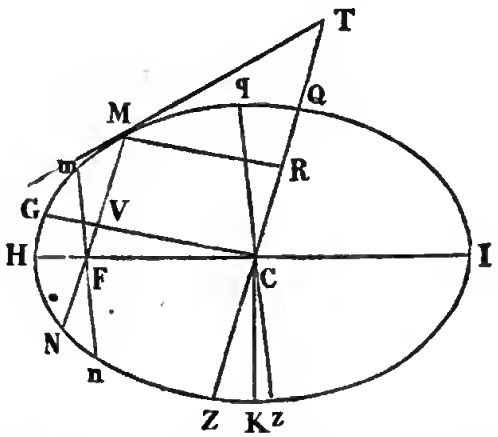


Draw CG perpendicular to PT and CM parallel to PF, and let CK be the conjugate semi-axis; then on account of the parallel lines, the right angled triangles CGM, DEP are equiangular, and $CM : CG :: DP : PE$, ther. $CM \times PE = CG \times DP = CK^2$ (Prop. XXVII.) But (Cor. Prop. XXV.) $CM = CI$,

ther. $CI \times PE = CK^2$; or, $CI : CK :: CK : PE$=half the latus-rectum. (Def. 18.)

Prop. XXX.

If H I be the transverse axis, C the center, Q Z any diameter, and M N parallel to it, drawn through the focus F; then $HI \times MN = QZ^2$.



Draw the diameter C G conjugate to Q Z, which will bisect M N in V. (Prop. XV.) Draw M R an ordinate to Q Z and the tangent M T meeting Q Z in T. Because C T is parallel to F M, C T is equal to C I (Cor. Prop. XXV.) and $CT \times CR = CQ^2$. (Prop. XVII.) Ther. $CI \times VM = CQ^2$, or $2CI \times 2VM = 4CQ^2$; that is, $HI \times MN = QZ^2$.

Cor. 1. If m n be any other right line drawn through the focus F, and the diameter q z parallel thereto, then $MN : mn :: CQ^2 : Cq^2$. For $HI \times mn = Cq^2$ (by the Prop.). But H I is given, therefore, $MN : mn :: CQ^2 : Cq^2$.

Cor. 2. $MF \times FN : mF \times Fn :: MN : mn$.

For (Cor. 4. Prop. XIV.) $MF \times FN : mF \times Fn :: CQ^2 : Cq^2$, but (Cor. 1.) $MN : mn :: CQ^2 : Cq^2$.

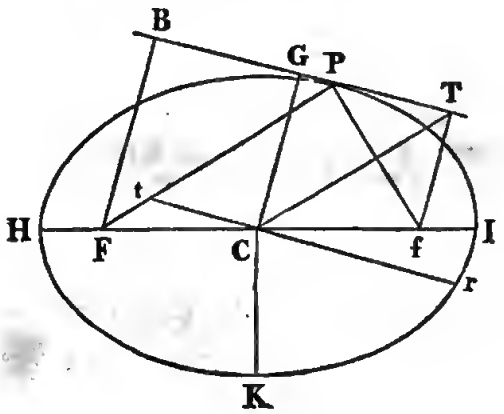
Therefore $MF \times FN : mF \times Fn :: MN : mn$.

Cor. 3. $\frac{1}{FN} + \frac{1}{FM} = \frac{2}{\text{latus-rectum}}$. For (Cor. 2.) $MF \times FN : MF + FN :: HF \times FI : HI$.

Ther. $\frac{MF+FN}{MF \times FN} = \frac{HI}{HF \times FI} = \frac{HI}{CK^2}$ (Cor. 6. Def.) $= \frac{1}{FN} + \frac{1}{FM} = \frac{2}{\text{latus-rectum}}$, CK being the semi-conjugate.

Prop. XXXI.

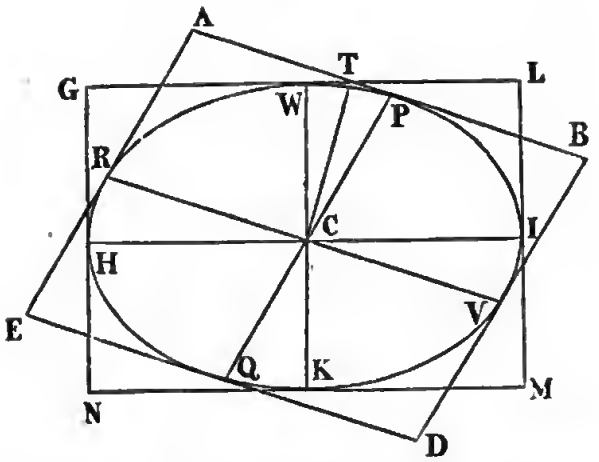
If HI be the transverse and CR CK the semi-conjugate axis, F, f, the foci, FB drawn perpendicular to the tangent PT, FP joined and Cr drawn parallel to PT, then FB : FP :: CK : Cr.



Draw fT perpendicular to PT and join fP; because the angles fPT, FPB are equal (Cor. Prop. XI.) the right angled triangles fTP, FBP are similar, and FB : FP :: fT : fP, or, $FB^2 : FB \times fT :: FP^2 : FP \times fP$, that is, $FB^2 : FP^2 :: CK^2 : Cr^2$. (Prop. XXIV. and XXVI.) Ther. FB : FP :: CK : Cr.

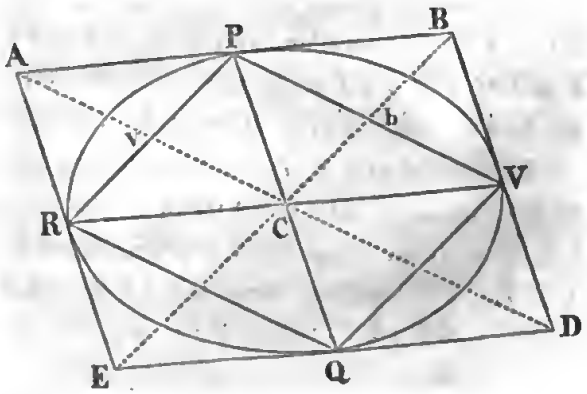
Cor. 1. If from the center C, CG be drawn perpendicular to the tangent PT, then $CG \times Cr = CI \times CK$. For draw CT parallel to FP, then (Cor. Prop. XXV.) CT=Pt=CI, and the triangles GCT, BFP similar; ther. CG : CT :: FB : FP :: CK : Cr; that is, CG : CI :: CK : Cr; or, $CG \times Cr = CI \times CK$.

Cor. 2. If a parallelogram ABDE be formed by drawing tangents through the vertices of any two conjugate diameters PQ, RV, it will be equal to the rectangle under the axes.



Let C be the center, H I, K W the axes, and R V, P Q the conjugate diameters; the tangents drawn through the vertices P V Q R will form a parallelogram. (Cor. Prop. XII.) Let the tangents drawn through P and V meet at B; P C V B will then be a fourth part of the parallelogram A B D E; draw C T perpendicular to A B or R V, then (Cor. 1.) C T × C V = C I × C K; that is, the parallelogram P C V B = C I K M; ther. A B D E = G L M N.

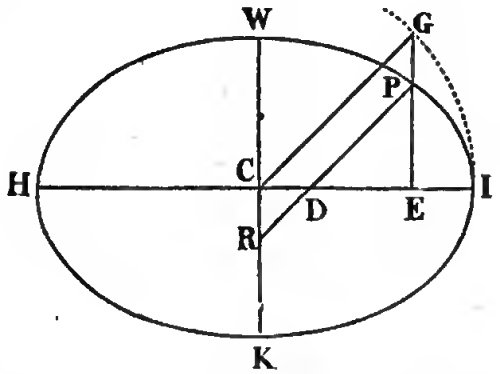
Cor. 3. The inscribed parallelogram Q R P V is half the circumscribed parallelogram A B D E.



By drawing the diagonals A D, B E, it will plainly appear that Q R P V is a *parallelogram*, and that the parallelogram P v C b is *half* the parallelogram P C V B, whence the Cor. is evident.

Prop. XXXII.

If H I be the transverse, K W the conjugate axis, and C the center, and if the difference between C I and C K be set from a point R in the conjugate to meet the transverse in D, and R D produced to P, so that D P be equal to C K, the point P will be in the ellipse.



Draw P E an ordinate to the transverse axis, and C G parallel to R D to meet E P in G; because C R, P G are parallel, C G is equal to R P: that is (by construction) to C I; therefore the point G is in the circumference of a circle described about H I; and because the triangles P E D, G E C are similar, $PE^2 : GE^2 :: PD^2 : GC^2$; that is, $PE^2 : HE \times EI :: CK^2 : CI^2$; therefore the point P is in the ellipse (Prop. XVI.)

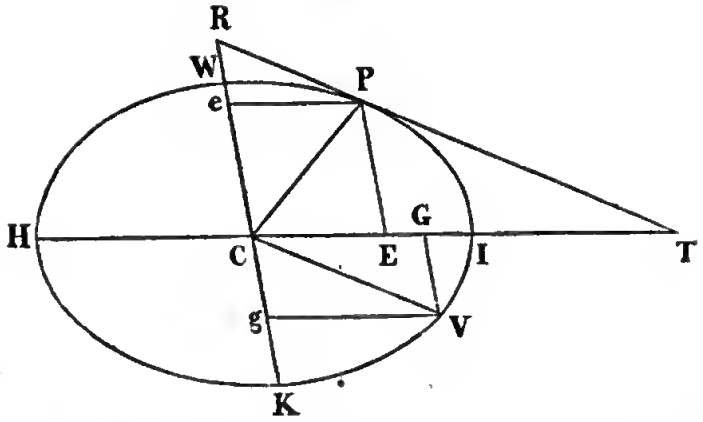
Cor. If two right lines H I, K W bisect each other at right angles in the point C, and R D = C I − C K, and D P = C K; and the line R D be moved through four right angles, so that the point R be always in K W, and D in H I, the point P will describe an ellipse whose transverse axis is H I, and whose conjugate is K W.

Hence the construction of the *elliptical compass*, or *trammel* is evident.

Prop. XXXIII.

If C P, C V be two conjugate diameters, C the center, and the ordinates P E, V G be drawn to a third diameter H I,

Then $HG \times GI = CE^2$; and $HE \times EI = CG^2$.



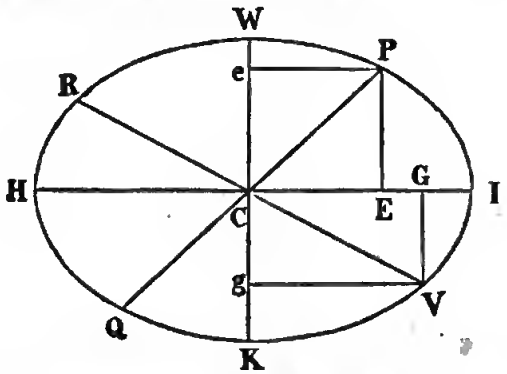
For draw the tangent P T meeting C W in R, then because P T is parallel to C V and P E to V G, the triangles P E T, V G C are equiangular; and therefore R P : P T :: C E : E T. Or, R P × P T : C E × E T :: P T² : E T² :: C V² : C G²; but (Prop. XXII.) R P × P T = C V², therefore C E × E T = C G² = H E × E I (Cor. 1. Prop. XVII.) = C I² − C E². And C I² − C G² = H G × G I = C I² − C I² + C E² = C E².

Cor. 1. C E² + C G² = C I². For C E² + C G² = H G × G I + H E × E I = C I², E I² − C G² + C I² − C E²; ther. C E² + C G² = C I².

Cor. 2. E P² + G V² = C W². For by drawing the ordinates P e, V g; C e² + C g² = E P² + G V² = C W².

Prop. XXXIV.

The sum of the squares of any two conjugate diameters is equal to the sum of the squares of the axis.

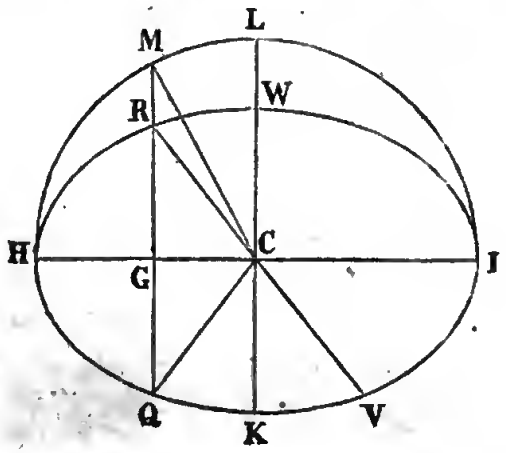


Let C I, C W be the semi-axes, and C P, C V two conjugate

semi-diameters, and let P E, V G be perpendicular to C I, and P e, V g perpendicular to C W. Then (Cor. 1 and 2. Prop. XXXIII.) $C I^2 = C E^2 + C G^2$, and $C W^2 = C e^2 + C g^2 = P E^2 + V G^2$; therefore $C I^2 + C W^2 = C E^2 + C G^2 + P E^2 + V G^2 = C P^2 + C V^2$ (47. 1. E.) Ther. $H I^2 + K W^2 = P Q^2 + V R^2$.

Prop. XXXV.

The nearer the diameters are to the transverse axis, the greater they are; and the more remote, the less they are. The conjugate axis is the least of all the diameters; and any two diameters, which make equal angles with either axis, are equal.



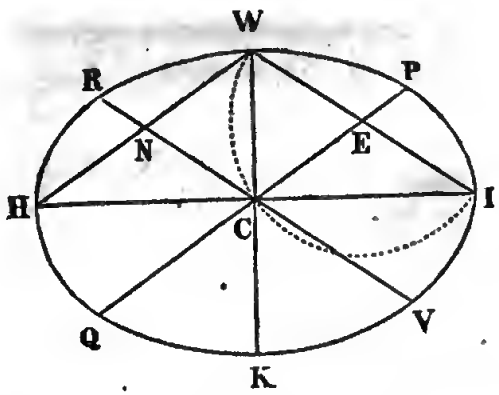
Describe a circle on the transverse axis H I, and let the ordinate G R be produced to meet the circumference of the circle in M. Then (Prop. VII.) $M G^2 : R G^2 :: C L^2 : C W^2$, therefore $M G^2 - R G^2 : M G^2 :: C L^2 - C W^2 : C L^2$; but $C M^2 - C G^2 = M G^2$ and $C R^2 - C G^2 = R G^2$; ther. $M G^2 - R G^2 = C M^2 - C R^2 = C I^2 - C R^2$. Ther. $C I^2 - C R^2 : M G^2 :: C I^2 - C W^2 : C I^2$; from whence $C R^2 = C I^2 + \left(\frac{C W^2}{C I^2} - 1\right) \times M G^2$. Ther. if the line M R G move from H to C, C R will decrease while M G increases, (for the quantity $\frac{C W^2}{C I^2} - 1$ is negative) and when M G becomes L C or C I, then $C R^2 = C W^2$; or, C R = C W, when it will be the least.

Secondly. Let R C V be any diameter, and R G Q an ordinate to either axis, and join C Q. Then G R being equal to G Q, and C G common to the two right angled triangles C G R, C G Q, the triangles are equal; ther. C Q=C R, and the angle Q C G= R C G.

Cor. Hence if the semi-diameters C R, C Q be equal, and the angle R C Q bisected by the line C H, the position of the axis will be determined.

Prop. XXXVI.

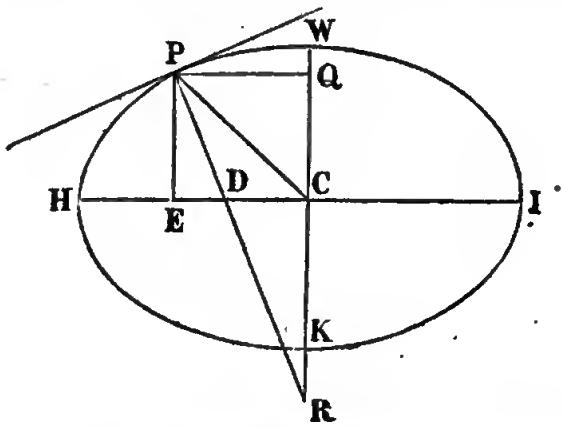
The two diameters which bisect the right lines joining the vertices of the axes, are equal and conjugate diameters.



Let H I, K W be the axes; join I W, W H, and draw the diameters P Q, R V, bisecting the lines I W, W H, in E and N. Because I W, W H are bisected by the diameters P Q, R V, they are ordinates to these diameters. (Prop. XV.) And because H I is bisected in C, and I W in E, C E is parallel to H W; therefore P Q, R V are conjugate diameters; and the angle I C W being a right angle, it will be in a semicircle, of which I W is the diameter, and E the center; therefore E I will be equal to E C, and the angle E C I equal to the angle E I C, which is equal to the alternate angle I C V; and therefore the diameters P Q, R V are equal. (Prop. XXXV.)

Prop. XXXVII.

Given the semi-transverse and conjugate diameters C H, C K, and the angles P D H, P C H, to find the semi-diameter C P, the ordinates P Q, P E, the normals P D, P R, and C D.



To find the Semidiameter C P.

By Prop. III. $CH^2 - CE^2 : PE^2 :: CH^2 : CK^2$.

And per trig. $PE^2 : CE^2 ::$ tang.^{2} E C P : Radius.

Therefore - - $CH^2 - CE^2 : CE^2 :: t^2 \times CH^2 : CK^2$.

Or - - - - - - $CH^2 : CE^2 :: t^2 \cdot CH^2 + CK^2 : CK^2$.

Per trig. - - - $CE^2 : c^2 :: CP^2 : 1$.

Therefore - - $CH^2 : c^2 :: CP^2 (t^2 \cdot CH^2 + CK^2) : CK^2$.

Hence $CP^2 = \frac{CH^2 \times CK^2}{c^2(CK^2 + t^2 \cdot CH^2)}$ $\therefore CP = \frac{CH \times CK}{c\sqrt{CK^2 + t^2 \cdot CH^2}}$

c = cos. and t = tang. E C P.

To find the Ordinate P Q = C E.

Put $CH^2 = m^2$, $CK^2 = n^2$ Tang. E D P = T, Tang. E C P = x.

Then by Cor. 4. Prop. XXVIII. Tang. E D P : Tang. E C P :: $CH^2 : CK^2$.

That is, $T : x :: m^2 : n^2$ $\therefore \frac{n^4}{m^4} \times T^2 = x^2$.

Per trig. Sec. E C P : Rad. :: C P : C E.

That is, $1 + x^2 : 1 :: CP^2 : CE^2$ $\therefore CE^2 = \frac{1}{1+x^2} \times CP^2$.

But $CP^2 = \frac{m^2 n^2}{c^2(n^2 + x^2 m^2)}$; ($c$ = cos. ECP).

Ther. $CE^2 = \frac{m^2 n^2}{(1+x^2)(n^2+x^2m^2)c^2} = \frac{m^2 n^2}{n^2 + x^2 m^2}$; $(1+x^2)c^2 =$ rad.^{2} by substituting for x^2, we have $CE^2 = \frac{m^2 n^2}{n^2 + \frac{T^2 n^4}{m^2}} = \frac{m^4 n^2}{m^2 n^2 + n^4} = \frac{m^4}{m^2 + T^2 n^2}$ Ther. $CE = PQ = \frac{CH^2}{\sqrt{CH^2 + T^2 . CK^2}}$.

To find the Normal PR.

Per trig. Sin. PRQ : PQ :: Rad. : PR.

Or, $C : PQ :: 1 : PR = \frac{PQ}{C} = \frac{CH^2}{C\sqrt{CH^2 + T^2 . CK^2}}$. ($C$ = cos. EDP).

To find the Normal PD.

By Cor. 1. Prop. XXVIII. we have $EC : ED :: CH^2 : CK^2$.

Ther. $ED^2 = \frac{CE^2 \times CK^4}{CH^4} = \frac{n^4}{m^4} \times CE^2$, and per trig. $C^2 : ED^2 :: 1 : PD^2 = \frac{ED^2}{C^2} = \frac{n^4}{C^2 m^4} \times CE^2 = \frac{n^4}{C^2 m^4} \times \frac{m^4}{m^2 + T^2 n^2} = \frac{n^4}{C^2(m^2 + T^2 n^2)}$ $\therefore PD = \frac{CK^2}{C\sqrt{CH^2 + T^2 . CK^2}}$.

To find the Ordinate PE.

Per trig. Rad. : PD :: Sin. EDP : PE $\therefore PE^2 =$ Sin.^{2} EDP $= PD^2$.

That is, $PE^2 = \frac{S^2 . n^4}{C^2(m^2 + T^2 n^2)} = \frac{T^2 n^4}{m^2 + T^2 n^2} = \frac{n^4}{n^2 + \frac{m^2}{T^2}} = \frac{n^4}{n^2 + {}^vT^2 m^2}$.

Ther. $PE = \frac{CK^2}{\sqrt{CK^2 + {}^vT^2 . CH^2}}$; (S=sin. and ${}^vT$=cot. EDP).

To find the Distance CD.

By Cor. 1. Prop. XXVIII. we have $EC : ED :: CH^2 : CK^2$.

Ther. $EC-ED : EC :: CH^2-CK^2 : CH^2 \therefore CD = \frac{CE}{CH^2}(CH^2-CK^2)$.

That is, $CD = \frac{CH^2-CK^2}{\sqrt{CH^2-T^2.CK^2}}$.

Prop. XXXVIII.

If the ellipse be nearly a circle, to find the variation of CP.

By Prop. XXXVII. $CP^2 = \frac{m^2n^2}{c^2(n^2+t^2m^2)}$* $= \frac{m^2n^2}{c^2n^2+s^2m^2}$

$= \frac{m^2n^2}{(1-s^2)n^2+s^2m^2} = \frac{m^2n^2}{n^2-n^2s^2+m^2s^2} = \frac{m^2n^2}{n^2+(m^2-n^2)s^2}$

$= \frac{m^2n^2}{n^2+(m+n)(m-n)s^2} = \frac{m^2n^2}{n^2+2n(m-n)s^2}$ nearly, $= \frac{m^2n^2}{\{n+(m-n)s^2\}^2}$ nearly,

Therefore $CP = \frac{mn}{n+(m-n)s^2}$ nearly.

From hence $m-CP = \frac{CP}{n}(m-n)s^2 = (m-n)s^2$ nearly,

And by substituting $1-c^2$ for s^2 we have $CP-n = (m-n)c^2$ nearly; and by restoring the values of m, n, s and c, we obtain,

$CH-CP = (CH-CK)\times \text{Sin.}^2 ECP$
$CP-CK = (CH-CK)\times \text{Cos.}^2 ECP$
} But CH−CK is given, therefore

CH−CP is as $\text{Sin.}^2 ECP$; and CP−CK is as $\text{Cos.}^2 ECP$ nearly.

Cor. Since $CP-CK : CH-CP :: \text{cos.}^2 ECP : \text{sin.}^2 ECP :: 1 : \frac{\text{sin.}^2}{\text{cos.}^2} ECP :: 1 : \text{tang.}^2 ECP$ (rad.=1).

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* $CP = \frac{mn}{\sqrt{n^2+(m^2-n^2)s^2}} = \frac{mn}{n+\frac{(m^2-n^2)s^2}{2n}}$ nearly, $= \frac{2mn^2}{2n^2+(m+n)(m-n)s^2}$

$= \frac{2mn^2}{2n^2+2n(m-n)s^2} = \frac{mn}{n+(m-n)s^2}$, as before.

It appears that the difference between C P and C K is *greater* than the difference between C H and C P, whilst the angle E C P is less than 45 degrees.

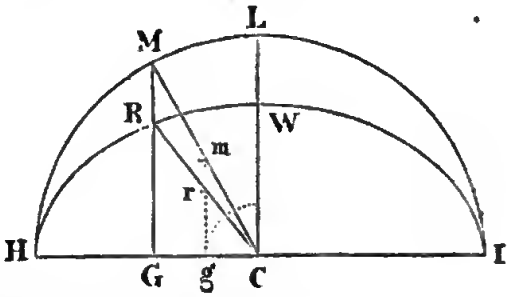
At 45 degrees the differences are *equal*.

And above 45 degrees, the former difference is *less* than the latter.

At 45°, $CP-CK=CH-CP \therefore CP=\frac{CH+CK}{2}$.

Prop. XXXIX.

If on the transverse diameter H I, a semicircle H L I be described, and the ordinate G R drawn and produced to cut the circle in M, and C R, C M joined, it is required to find the angle R C M.



Put g m, the tang. G C M=T
And g r, the tang. G C R=t } Rad. c g=I.

Then (Prop. VII.) G M : G R :: C L :: C W : g m : g r :: T : t $\therefore t=\frac{CW \times T}{CI}$; and per trig. $\left(T-\frac{CW \times T}{CI}\right) \div \left(1+\frac{CW \cdot T^2}{CI}\right)=\frac{(CI-CW) \times T}{CI+CW \cdot T^2}=$ tang. R C M.

To find when the Angle R C M *will be a Maximum.*

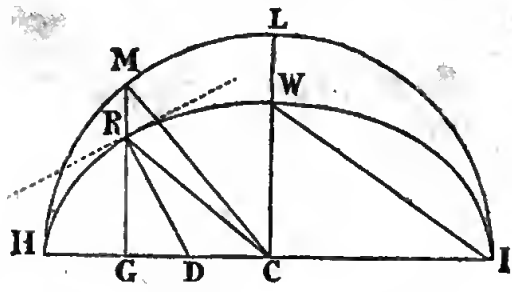
Or $\frac{T}{CI+CW \cdot T^2}=$ max. $=\frac{1}{\frac{CI}{T}+CW \cdot T}$, hence we have to find $\frac{CI}{T}+CW \cdot T=$max. In order to which we must find when

the sum of the tangent and co-tangent of any angle will be the *least possible,* or $T+\frac{1}{T}=\frac{1+T^2}{T}=(1+T^2)\times\frac{1}{T}=\frac{1}{\cos.^2}\times\frac{\cos.}{\sin.}$ $=\frac{1}{\text{Sin.}\times\text{Cos. G C M}}=\text{min.}$ Therefore $T+\frac{1}{T}$ is the least, when the sin. × Cos. is the greatest, that is, when the Sin.=Cos., or the angle G C M *forty-five degrees,* in which case T=1, therefore the tang.=cotang. Hence $\frac{CI}{T}=CW\times T$, ther. $T^2=\frac{CI}{CW}$ = tang. C W I, (by joining I W), and by substitution we have $\frac{(CI-CW)\times T}{CI+CW\cdot T^2}=\frac{CI-CW}{2CI\times CW}\times\sqrt{CI\times CW}=$ tang. R C M when a maximum.

If the ellipse do not differ much from a circle, (as is the case with our earth) the tang. R C M would be very nearly equal to $\frac{CI-CW}{2CI}=\frac{WL}{HI}=\frac{\text{diff. of semi-diam.}}{\text{transverse axis}}$ when a maximum. Or by making T=1, we obtain tang. R C M $=\frac{CI-CW}{CI+CW}=\frac{WL}{HI}$ nearly.

Prop. XL.

Let the figure in the margin be similar to that of Prop. XXXIX, and draw the normal R D, it is required to find the angle D R C, the difference between G R C and G R D.



Per Cor. 4. Prop. XXVIII. we have G C : G D :: tang. G R C : tang. G R D :: C I^2 : C W^2. Ther. tang. G R C × C W^2=tang. G R D × C I^2. Put tang. G R C=X, and tang. G R D=Y; then

$X = \frac{CI^2}{CW^2} \times Y$. And per trig. $\left(\frac{CI^2}{CW^2} \times Y - Y\right) \div \left(1 + \frac{CI^2 \cdot Y^2}{CW^2}\right)$ $= \text{tang. } DRC = \frac{(CI^2 - CW^2) \times Y}{CW^2 + CI^2 \cdot Y^2}$. But tang. $GRD = Y = \frac{CW^2}{CI^2} \times \text{tang. } GRC = \frac{CW^2}{CI^2} \times \text{cotang. } GCR$. But (Prop. XXXVIII.) cot. $GCR = \frac{I}{t} = \frac{CI}{CW} \times \frac{I}{T}$ (T=tang. GCM).

Therefore $Y = \frac{CW^2}{CI^2} \times \frac{CI}{CW} \times \frac{I}{T} = \frac{CW}{CI} \times \frac{I}{T}$; and by substituting for Y and Y^2 we have tang. $DRC = \frac{(CI^2 - CW^2) \times \frac{CW}{CI} \times \frac{I}{T}}{CW^2 + CI^2 \times \left(\frac{CW}{CI} \times \frac{I}{T}\right)^2} = \frac{CI^2 - CW^2}{CI \times CW} \times \frac{T}{I + T^2} = \frac{CI^2 - CW^2}{CI \times CW}$ $\times$ Sin. $\times$ Cos. GCM.

Therefore the tang. DRC varies as Sin. $\times$ Cos. GCM, and will be a maximum when Sin. and Cos. GCM are equal, or GCM 45 degrees, or when T=1. According to which,

The tang. $GRD = Y = \frac{CW}{CI} \times \frac{I}{T} = \frac{CW}{CI} = \text{tang. } CIW$,

And tang. $GRC = X = \frac{CI^2}{CW^2} \times Y = \frac{CI}{CW} = \text{tang. } CWI$,

But $\frac{CW}{CI} \times \frac{CI}{CW} = 1$. Therefore the angles GRC, GRD, are complements to each other, and IW parallel to CR, when the angle DRC is greatest. And by substituting for T, in the equation tang. $DRC = \frac{CI^2 - CW^2}{CI \times CW} \times \frac{T}{I + T^2}$, we have tang. $DRC = \frac{CI^2 - CW^2}{2CI \times CW}$.

That is, $2CI \times CW : (CI + CW)(CI - CW) :: \text{Rad.} : \text{Tang. } DRC$.

If the ellipse be nearly a circle, then 2 CI or 2 CW, would be nearly equal to CI+CW, in which case, tang. $DRC = \frac{CI - CW}{CI}$ $\frac{WL}{\frac{1}{2}HI}$ very nearly when the angle DRC is a maximum.

Hence the angle DRC when greatest may be thus found:

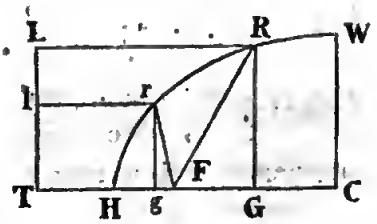
Join IW, draw CR parallel thereto, and draw the normal RD; the angle DRC thus found will be the greatest.

By Prop. XXXIX. tang. $RCM = \frac{WL}{HI}$, and by this Prop. tang. $RDC = \frac{WL}{\frac{1}{2}HI}$, nearly, in the case of a maximum. Hence tang. $RCM = \frac{1}{2}$ tang. DRC.

Prop. XLI.

Let HC and CW be the semi-axes, TL the directrix, C the center, F the focus, and R any point in the curve.

Then $FR = \frac{\text{half latus-rectum}}{1 - \frac{CF}{CH} \times \text{cos. } RFG}$.



Draw the ordinate RG, then per trig. Rad. : FR :: cos. RFG : FG = FR × C (C = cos. RFG) and (Cor. 2. Prop. IX.) FR : RL :: CF : CH. But RL = GF + FT, ther. FR : GF + FT :: CF : CH, and FR × CH = CF × FG + CF × FT = CF × FG + CW^2 (Cor. 4. Prop. IX.) that is, FR × CH = CW^2 + CF × FR × C, therefore FR × CH − FR × CF × C = CW^2, or (CH − CF × C) × FR = CW^2 ∴ $FR = \frac{CW^2}{CH - CF \times C} = \frac{CW^2}{CH} \times \frac{1}{1 - \frac{CF}{CH} \times C}$

$= \frac{\text{half latus-rectum}}{1 - \frac{CF}{CH} \times \text{cos. } RFG}$.

If rF make an obtuse angle rFC, then Fr × CH = CW^2 − CF × Fr × C.

And thence, $Fr = \frac{\text{half latus-rectum}}{1 + \frac{CF}{CH} \times \text{cos. } rFg}$

Note to Prop. XLI.

If C H $=1$, C W $=c$, $\angle$ C F R $=z$, C F $=w$: Then $FR=\frac{CW^2}{CH-CF\times C}=\frac{1}{1-w\cos.z}\times c^2$ = (by division) $c^2(1+w\cos.z+w^2\cos.^2z+w^3\cos.^3z+$&c.) By trig. $\cos.^2z=\frac{1}{2}+\frac{1}{2}\cos.2z$, $\cos.^3z=\frac{3}{4}\cos.z+\frac{1}{4}\cos.3z$, and (Prop. VI.) $c^2=1-w^2$, therefore by substitution we have

$$FR=c^2(1+\tfrac{1}{2}w^2+w\cos.z+\tfrac{3}{4}w^3\cos.z+\tfrac{1}{2}w^2\cos.2z+\tfrac{1}{4}w^3\cos.3z)$$

$$=(1-w^2)\{1+\tfrac{1}{2}w^2+(w+\tfrac{3}{4}w^3)\cos.z+\tfrac{1}{2}w^2\cos.2z+\tfrac{1}{4}w^3\cos.3z.\}$$

$$=1-\tfrac{1}{2}w^2-\tfrac{1}{2}w^4+(w-\tfrac{1}{4}w^3-\tfrac{3}{4}w^5)\cos.z+(\tfrac{1}{2}w^2-\tfrac{1}{2}w^4)\cos.2z+(\tfrac{1}{4}w^3-\tfrac{1}{4}w^5)\cos.3z.$$

By omitting the terms affected with w^4 and w^5,

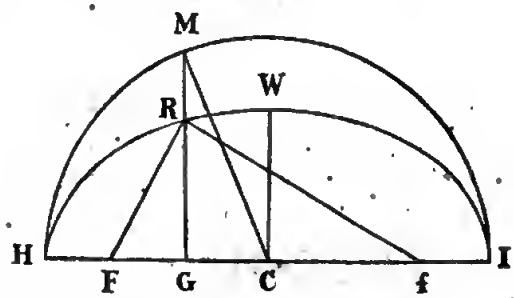
$$FR=1-\tfrac{1}{2}w^2+(w-\tfrac{1}{4}w^3)\cos.z+\tfrac{1}{2}w^2\cos.2z+\tfrac{1}{4}w^3\cos.3z.$$

If the eccentricity C F be very small, then $c^2=1-w^2=1$, nearly; and therefore all the terms connected with w^2, w^3, &c. may be neglected; then $FR=1+w\times\cos.z$, and $FR-1=w\times\cos.z$; hence the variation of F R is nearly in proportion to cos. C F R.

Prop. XLII.

On the transverse axis H I describe a semicircle H M I, and draw the ordinate R G, which produce to M; join M C, and draw R F, R f to the foci.

Then $FR=CH-CF\times\cos.MCH$.



By Euclid, Book ii. Prop. 13. $FR^2=Ff^2+fR^2-2Ff\times fG$.

Ther. $fR^2-FR^2=2Ff\times fG-Ff^2$.

But $fR^2-FR^2=(fR+FR)(fR-FR)=2HC\times(CH-FR)2$.

And $2fF \times fG - Ff^2 = (2fG - Ff) \times Ff = (2fG - 2CF) \times 2CF = 2CG \times 2CF$.

Ther. $2HC \times (CH - FR) \times 2 = 2CG \times 2CF$; or, $CM \times (CH - FR) = CF \times CG$.

Per trig. $GC = CM \times \text{cos. } GCM$, ther. $CH - FR = CT \times \text{cos. } GCM$, and $FR = CH - CF \times \text{cos. } HCM$. Ther. $Rf = CH + CF \times \text{cos. } MCH$.

And by Prop. XLI. - - $Rf = \frac{CW^2}{CH} \times \frac{1}{1 - \frac{CF}{CH} \times \text{cos. } RfH}$.

Cor. 1. The sin. RfH : sin. MCH :: CW : fR.

For, Rad. : Rf :: sin. RfG : $RG = Rf \times \text{sin. } RfG$.

And, CI : CW :: GM : $RG = \frac{CW}{CI} \times GM$.

But, Rad. : CM :: sin. GCM : $GM = CI \times \text{sin. } GCM$.

Ther. $\frac{CW}{CI} \times CI \times \text{sin. } MCH = Rf \times \text{sin. } RfH$.

Whence sin. RfH : sin. MCH :: CW : fR.

Cor. 2. Cos. $RfH = \frac{CF + CH \times \text{cos. } MCH}{CH + CF \times \text{cos. } MCH}$, and

Cos. $MCH = \frac{CH \times \text{cos. } RfH - CF}{CH - CF \times \text{cos. } RfH}$.

For by equating the expressions for Rf. Prop. XLI. XLII.

$CH + CF \times \text{cos. } MCH = \frac{CW^2 = CH^2 - CF^2}{CH - CF \times \text{cos. } RFH}$; from whence the values of cos. RfH, and cos. MCH, as stated in this corollary.

Cor. 3. As sin. $\frac{RfH}{2}$: sin. $\frac{MCH}{2}$ :: $\sqrt{fI}$: $\sqrt{fR}$.

Put $CH = m$, $CF = n$, Cos. $MCH = C$, Cos. $RfH = c$. Then per Cor. 2. $c = \frac{n + mC}{m + nC}$, $C = \frac{mc - n}{m - nc}$. And per Emer. Trig. page 12.

Sin. $\frac{RfH}{2} = \sqrt{\frac{1 - c}{2}} = \sqrt{\left(1 - \frac{n + mC}{m + nC}\right) \times \frac{1}{2}} = \sqrt{\frac{m + nC - n - mC}{2 \times (m + nC)}}$

$$=\sqrt{\frac{(m-n)\,(1-C)}{2Rf}}=\sqrt{\frac{1-C}{2}}\times\sqrt{\frac{HF}{Rf}}=\text{sin.}\,\frac{MCH}{2}\times\frac{\sqrt{HF}}{\sqrt{Rf}}.$$

Hence we have, $\text{sin.}\,\frac{RfH}{2}:\text{sin.}\,\frac{MCH}{2}::\frac{\sqrt{HF}}{\sqrt{Rf}}:1::\sqrt{HF}:\sqrt{Rf}::\sqrt{fI}:\sqrt{fR}.$

Cor. 4. $FI:CW::\text{Tang.}\,\frac{MCH}{2}:\text{Tang.}\,\frac{RfH}{2}.$

Per trig. ib. $\text{Tang.}\,\frac{RfH}{2}=\sqrt{\frac{1-c}{1+c}}.$

$$1-c=1-\frac{n+mC}{m+nC}=\frac{m+nC-n-mC}{m+nC}=\frac{(m-n)\times(1-C)}{m+nC}=\frac{HF\times(1-C)}{Rf},$$

$$1+c=1+\frac{n+mC}{m+nC}=\frac{m+nC+n+mC}{m+nC}=\frac{(m+n)\,(1+C)}{m+nC}=\frac{IF\times(1+C)}{Rf},$$

$$\frac{1-c}{1+c}=\frac{HF\times(1-C)}{Rf}\div\frac{IF\times(1+C)}{Rf}=\frac{1-C}{1+C}\times\frac{HF}{IF}=\frac{1-C}{1+C}\times\frac{HF\times IF}{IF^2}=\frac{1-C}{1+C}\times\frac{CW^2}{FI^2}.$$

Ther. $\sqrt{\frac{1-c}{1+c}}=\text{Tang.}\,\frac{RfH}{2}=\text{Tang.}\,\frac{MCH}{2}\times\frac{CW}{FI}$;

Ther. $1:\frac{CW}{FI}::FI:CW::\text{Tang.}\,\frac{MCH}{2}:\text{Tang.}\,\frac{RfH}{2}.$

Cor. 5. $\sqrt{IF}:\sqrt{FH}::\text{Tang.}\,\frac{MCH}{2}:\text{Tang.}\,\frac{RfH}{2}.$

For (Cor. 4.) $\sqrt{\frac{1-c}{1+c}}=\sqrt{\frac{1-C}{1+C}}\times\frac{\sqrt{HF}}{\sqrt{IF}}.$

Ther. $1:\frac{\sqrt{HF}}{\sqrt{IF}}::\sqrt{IF}:\sqrt{HF}::\sqrt{\frac{1-C}{1+C}}:\sqrt{\frac{1-c}{1+c}}::$
$\text{Tang.}\,\frac{MCH}{2}:\text{Tang.}\,\frac{RfH}{2}.$

Cor. 6. If $a=CH$, $ae=CF$, and $c=\text{cos. RFI}$.
Then, $FR:fR::1-e^2:1-2ec+e^2$.

For by Prop. XLI. $FR = \frac{CW^2 = CH^2 - CF^2}{CH - CF \times c} = \frac{a^2 - a^2e^2}{a - aec} = \frac{(1-e^2) \times a}{1-ec}$,

And $Rf = 2CH - FR = 2a - \frac{1-e^2}{1-ec} \times a = \frac{2-2ec-1+e^2}{1-ec} \times a = \frac{1-2ec+e^2}{1-ec} \times a$.

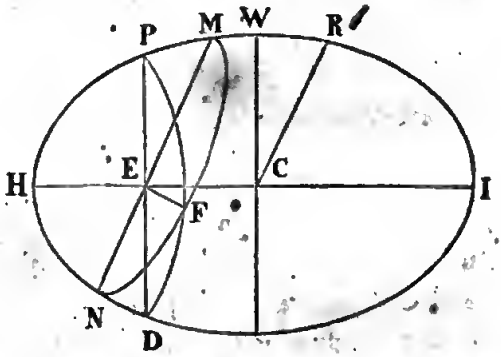
Ther. $\frac{FR}{fR} = \frac{1-e^2}{1-2ec+e^2}$; or, $FR : fR :: 1-e^2 : 1-2ec+e^2$.

Definition 22.

A *spheroid* is a solid conceived to be generated by the revolution of an ellipse about one of its axes, and is denominated either *prolate* or *oblate*, according to the revolution being made about the transverse or conjugate axis.

Prop. XLIII.

If a spheroid be cut by a plane in any position, except one perpendicular to the fixed axis, the section will be an ellipse.

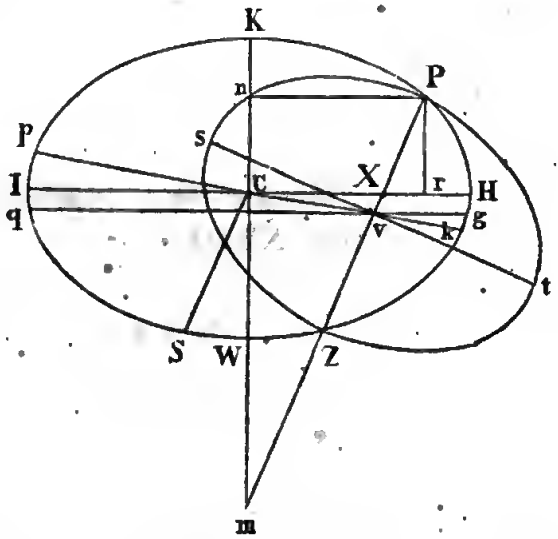


Let N M I be the generating section, or a section of the spheroid through its axis H I, and perpendicular to the proposed section M F N; their common intersection being M N; through the intersection E, draw P E D perpendicular to H I, and erect E F perpendicular to the plane N M I, and meeting the perpendicular plane in F. Then (Scholium to Prop. XIV.) we have $ME \times EN : PE \times ED :: CR^2 : CW^2$. But, by Definition 22, the points P E D will be in the circumference of a circle whose

diameter is PD, so that $PE \times ED = EF^2$; hence $ME \times EN : EF^2 :: CR^2 : CW^2$ (a constant ratio). Therefore (Prop. XVI.) MFN is an ellipse.

Prop. XLIV.

If IKHW be a section of an oblate spheroid through the transverse diameter IH, which is at right angles to the plane of the ellipse PsZt, formed by a plane passing through the common section PZ, required the transverse and conjugate diameters st and PZ.



Let PZ and st represent the principal axes of the elliptic section Pt Zs and v the center of PZ, through which draw gvq parallel to HI, and CS parallel to PZ; also demit the perpendiculars Pr, Pn, and produce PZ to meet KW in m. It appears from Definition 22, that qg is a diameter of a circle whose circumference passes through the four points q g s t, and therefore $qv \times vg = vs \times vt = (vs)^2 = (vt)^2$.

Put $CH^2 = m^2$, $CK^2 = n^2$, s=sin. c=cos. and t=tang. of the angle PXH=ICS. Then (Cor. 1. Prop. XXVIII.) we have $Cr : rX :: CH^2 : CK^2 \therefore Cr - rX : Cr :: CH^2 - CK^2 : CH^2$.

$$\text{Ther. } CX = \frac{Cr}{CH^2}(CH^2 - CK^2) = \frac{Cr}{m^2}(m^2 - n^2) = \frac{Cr \times d}{m^2}.$$

But (Prop. XXXVII.) $Pn = Cr = \frac{CH^2}{\sqrt{CH^2 + t^2 \cdot CK^2}} = \frac{m^2}{a}$

$PX = \frac{CK^2}{c\sqrt{CH^2 + t^2 \cdot CK^2}} = \frac{n^2}{ca}$, and $CS = \frac{CH \times CK}{c\sqrt{CK^2 + t^2 \cdot CH^2}} = \frac{mn}{cb}$.

By substituting for Cr in the value of CX, we have $CX = \frac{d}{a}$;

and $IX \times XH = CH^2 - CX^2 = m^2 - \frac{d^2}{a^2}$.

By Scholium to Prop. XIV.

$PX \times XZ : IX \times XH :: Pv \times vZ : qv \times vg :: CS^2 : CH^2$.

That is, $\frac{n^2}{ca} \times XZ : m^2 - \frac{d^2}{a^2} :: (Pv)^2 : (vs)^2 :: \frac{m^2 n^2}{b^2 c^2} : m^2$,

Ther. $Pv : vs :: \frac{n}{bc} : 1 :: n : bc$ $\therefore \frac{n^2}{ac} \times XZ = \frac{a^2 m^2 - d^2}{a^2} \times \frac{n^2}{b^2 c^2}$. Ther. $XZ = \frac{a^2 m^2 - d^2}{a c b^2}$ And $PX + XZ = PZ = 2Pv$,

that is, $\frac{n^2}{ac} + \frac{a m^2 - d^2}{a c b^2} = \frac{n^2 b^2 + a^2 m^2 - d^2}{a c b^2} = PZ = 2Pv$.

But $2Pv = \frac{2n \times vs}{bc} = \frac{n^2 b^2 + a^2 m^2 - d^2}{a c b^2}$ $\therefore 2vs = st = \frac{n^2 b^2 + a^2 m^2 - d^2}{a b n}$.

The latus-rectum (L) of the axis PZ may be thus found, $PZ : st :: st : L = \frac{n^2 b^2 + a^2 m^2 - d^2}{a n^2} \times c$ (Prop. V.)

By restoring the values of a, b, and d, we have

$$\begin{aligned} n^2 b^2 + a^2 m^2 - d^2 &= n^2 (n^2 + t^2 m^2) + m^2 (m^2 + t^2 n^2) - (m^2 - n^2)^2 \\ &= n^4 + m^2 n^2 t^2 + m^4 + m^2 n^2 t^2 - m^4 + 2 m^2 n^2 - n^4 \\ &= 2m^2 n^2 t^2 + 2 m^2 n^2 = (1 + t^2) \times 2 m^2 n^2 = \frac{2 m^2 n^2}{c^2}. \end{aligned}$$

From hence $PZ = \frac{2 m^2 n^2}{a b^2 c^2}$; $st = \frac{2 m^2 n^2}{a b n c^2} = \frac{2 m^2 n}{a b c^2}$.

And - - - - $L = \frac{2 m^2 n^2 \times c}{a n^2 c^2} = \frac{2 m^2}{a c}$; or thus,

Semi-axis $vs = \frac{CH^3 \times CK}{\sqrt{(CH^2+t^2 \cdot CK^2)}\sqrt{(CK^2+t^2 \cdot CH^2)}} \times \text{Sec.}^2 PXH.$

Semi-axis $vP = \frac{CH^3 \times CK^2}{(CK^2+t^2 \cdot CH^2)\sqrt{(CH^2+t^2 \cdot CK^2)}} \times \text{Sec.}^3 PXH.$

Half the latus-rectum of $PZ = \frac{CH^2}{\sqrt{(CH^2+t^2 \cdot CK^2)}} \times \text{Sec. } PXH$

$= Pm$ (Prop. XXXVII.) $=$ Rad. cur. at the vortex P of the section P t z s (Prop. VI. Cor. 2. cur.)

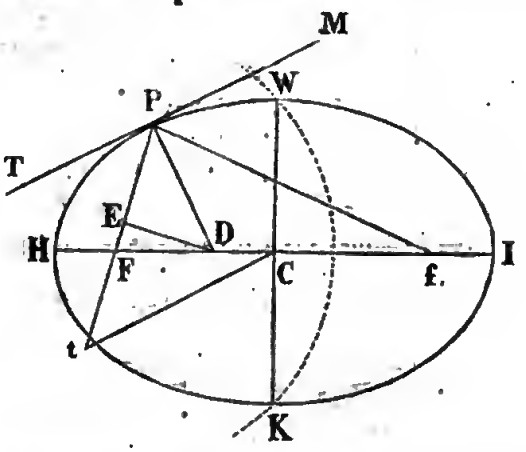
Prop. XLV.

If the diameter h p be drawn through the centers C and V, (fig. to Prop. XLIV.) the semi-diameter $Ch = \frac{CH \times Cv}{\sqrt{CH^2-(vs)^2}}$.

$Ch+Cv=vp$, $Ch-Cv=vh$; and per scholium, Prop. XIV. $vh \times vp : vg \times vq :: Ch^2 : CH^2$, or, $Ch^2-Cv^2 : Ch^2 :: (vs)^2 : CH^2$; therefore, $Cv^2 : Ch^2 :: CH^2-(vs)^2 : CH^2 \therefore Ch = \frac{CH \times Cv}{\sqrt{CH^2-(vs)^2}}$.

Prop. XLVI.

The distance of any point in an ellipse from the focus, the latus-rectum, and the position of the tangent at that point being given, to describe the ellipse.

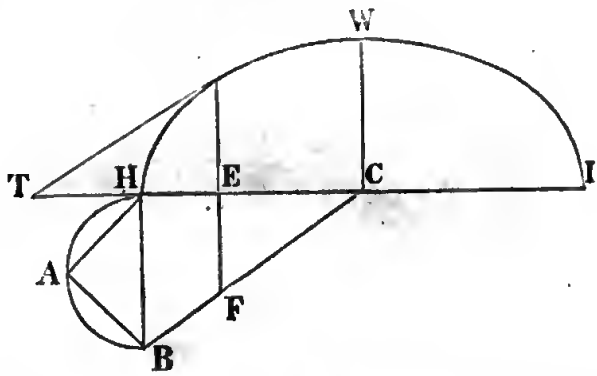


Let F be the focus, P the given point, and PT the tangent, making a given angle with PF.

Take P E=$\frac{1}{2}$ latus-rectum; draw P D perpendicular to P T, and E D perpendicular to F P; the point of intersection D is in the axis (Prop. XXIX.); join F D and produce it; make the angle M P f=angle T P f; and the point f in F D produced will be the other focus (Cor. 8. Prop. XI.); bisect F f in C, and draw C t parallel to P T, intersecting P F in t; make C H and C I=P t =half the transverse axis (Cor. Prop. XXV.); through C draw W K at right angles to H I; with the center F and distance P t, describe a circle intersecting K W in K and W, then K W will be the conjugate axes (Prop. VI.); hence we have the foci, and the transverse and conjugate axes given, to describe the ellipse.

Prop. XLVII.

If C H B be a triangle right angled at H, and the curve W P H such that drawing any right line F E P parallel to B H, it will always be $EF^2 + EP^2 = HB^2$; the curve W P H will be an *ellipse*, and C H, C W the *semi-axis*.



Let C I=C H. Then per similar triangles $CH^2 : HB^2 :: CE^2 : EF^2$. Ther. $CH^2 : HB^2 :: CH^2 - CE^2 : HB^2 - EF^2$; that is, $CH^2 : CW^2 :: EI \times EH : EP^2$, a property of the ellipse (Prop. XVI.)

Corollary 1.

If P T touches the ellipse at P, it will be $EF^2 : EP^2 :: CE : ET$. For (Prop. XVII.) C E : C H :: C H : C T. Ther. $CE^2 : CH^2 :: CE : CT$; and $CE^2 : CH^2 - CE^2 :: CE : CT - CE$; but $CE^2 : CH^2 - CE^2 :: EF^2 : HB^2 - EF^2$; therefore $EF^2 :$

$HB^2-EF^2 :: CE : CT-CE :: CE : ET$; that is, $EF^2 : EP^2 :: CE : ET$.

Corollary 2.

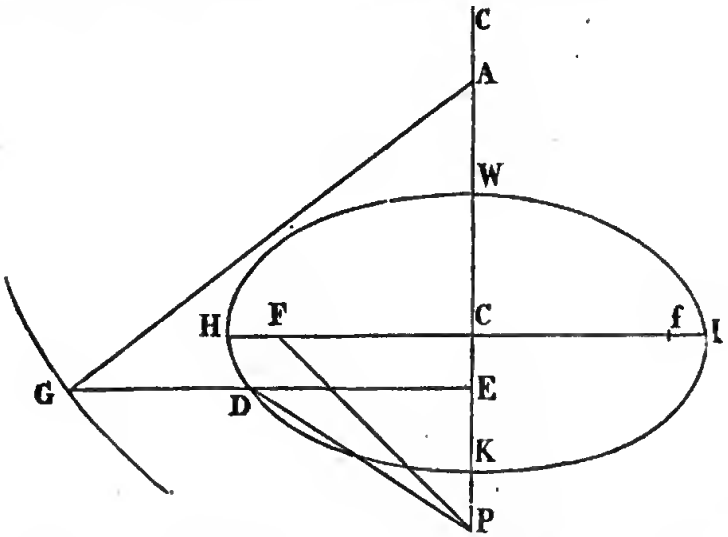
If the semicircle BAH be described on BH, the legs BA, AH of the right-angled triangle BAH, will be equal to the segments FE and EP of the right line FEP.

For $BA^2+AH^2=HB^2=EF^2+EP^2$.

Therefore $BA+AH=FE+EP=FP$.

Prop. XLVIII.

Let HIWK be an ellipse, HI the greater axis, WK the lesser, C the center, F, f, the foci, P a point in WK produced, PD a line meeting the ellipse in D, and EDG perpendicular to WK: Then if EG be so taken that FC : CK :: PD : EG, the locus of the point G, will be in the circumference of a circle, whose center is in the axis WK, or in WK produced.



Put $CH=t$, $CK=c$, $CF=m$, $CP=d$, and $CE=x$; then $WE=c+x$, $KE=c-x$, $PE^2=d^2-2dx+x^2$, and $PD^2=\frac{m^2}{c^2}\cdot EG^2$; also, Prop. VI. $m^2=t^2-c^2$, and Prop. III. $ED^2=t^2-\frac{t^2x^2}{c^2}$. Hence we have $PD^2=PE^2+ED^2=d^2+t^2-2dx+x^2-$

$\frac{t^2 x^2}{c^2}$. But $-\frac{m^2 x^2}{c^2} = x^2 - \frac{t^2 x^2}{c^2}$, ther.* $\frac{m^2}{c^2}$. $E G^2 = d^2 + t^2 - 2 d x - \frac{m^2}{c^2} x^2$; an equation of a circle, whose circumference is the locus of the point G, and whose center is in the line P W.

Cor. 1. If the point D be taken above the transverse axis H I, ther. $P E = d + x$ and $\frac{m^2}{c^2} E G^2 = d^2 + t^2 + 2 d x - \frac{m^2}{c^2} x^2$, an equation of a circle.

Cor. 2. If P falls on K or W, then $C P = C K$, or $b = d$, and $m^2 E G^2 = c^4 + c^2 t^2 \mp 2 c^3 x - m^2 x^2$.

Cor. 3. When P coincides with C, then $P C = d = o$, ther. $E G^2 = \frac{c^2 t^2}{m^2} - x^2$, an equation of a circle whose center is C, and radius $\frac{c t}{m}$.

Cor. 4. When $F C = C K$; then $E G = P D$.

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* Put $C E = x$, $E G = y$, and let $a \pm 2 b x - c x^2 = c y^2$, a, b, c, being constant quantities; make $A C = \frac{b}{c}$, and $A E = z$. We then have $A E \pm A C = C E = x = z \pm \frac{b}{c}$ from whence $2 b x = 2 b z \pm \frac{2 b^2}{c}$, and $c x^2 = c z^2 \pm 2 b z + \frac{b^2}{c}$, then by substitution, $\frac{a c + b^2}{c^2} - z^2 = y^2$, or $r^2 - A E^2 = E G^2$; an equation of a circle whose center is A, and semi-diameter $r = \frac{1}{c} \sqrt{a c + b^2}$.

OF THE

CIRCLE OF CURVATURE.

DEFINITION.

We know that a circle may be described through any three points taken in a given *curve;* and since an indefinite number of such points may be so taken, an indefinite number of circles may be described through those points; but the nearer those three points are together, the nearer will be the coincidence of the curve and circle. Therefore, as the *tangent* of a curve is the limit of all the straight lines which meet the curve in *two* points, so will the *osculating circle* be the limit of all the circles which pass through any *three* points in the given curve.

Hence we may conclude, that a circle of equal curvature with a curve at any given point therein, is that which passes through three points of the curve, which are supposed *indefinitely* near to each other; and therefore have the same common ordinate, chord, arch, and tangent, at the point of contact.

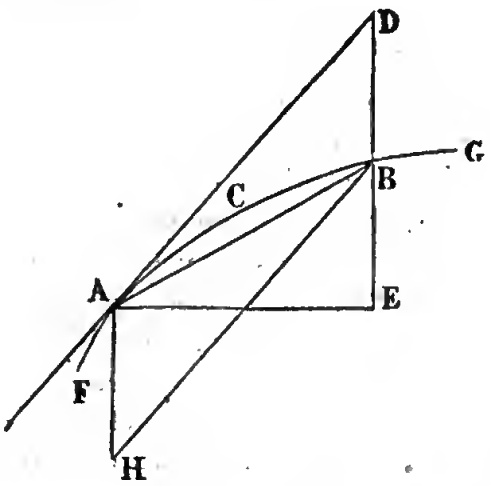
The following extract from Mr. Whiston's Euclid, page 68, is worthy of remark:—

1. Circles growing into amplitude greater than any given one, approach always, even unto infinity, nearer and nearer to the tangent, but are never joined to it, otherwise than in one single point of contact; which thing, although it be most evident, is yet truly admirable.

2. The angle of contact contained by the tangent and circumference, cannot be divided by any right line.
3. Nevertheless, by circumferences of circles touching the tangent in the same point, the angle of contact may be divided and diminished infinitely.
4. In the 1st and 3rd lies hid the whole mystery of asymptotes, that is, of a right line approaching unto an hyperbola, yet never concurring with it.

LEMMA.

The ultimate ratio of the arc A C B, chord A B, and tangent A D is that of equality.



Let A C B be any part of the curve F G, A B its chord, and A D a tangent at A. Through the point B draw the straight line D E, and let A E be at right angles thereto. Then in the right-angled triangles A E B, A E D, we have $AD^2 - (DB+BE)^2 = AB^2 - BE^2$; therefore $AD^2 - AB^2 = (DB+2BE)\,DB = (DE+BE)\,DB$; conceive the point B to move along the curve B C A towards the point A, and let D E be always parallel to its original position; on this supposition, D B, the subtense of the angle of contact, would be continually diminished, and if the point B actually coincided with the point A, D B would vanish,

and then $AD^2 - AB^2 =$ zero; therefore $\frac{AD}{AB} =$ unity, the limit of the ratio of AD to AB.

But the arc ACB being always within the triangle ABD, we may therefore conclude *a fortiori* that $\frac{ACB}{AB} =$ unity. Hence the ultimate ratio of the arc, chord, and tangent, is that of equality.

COROLLARY 1.

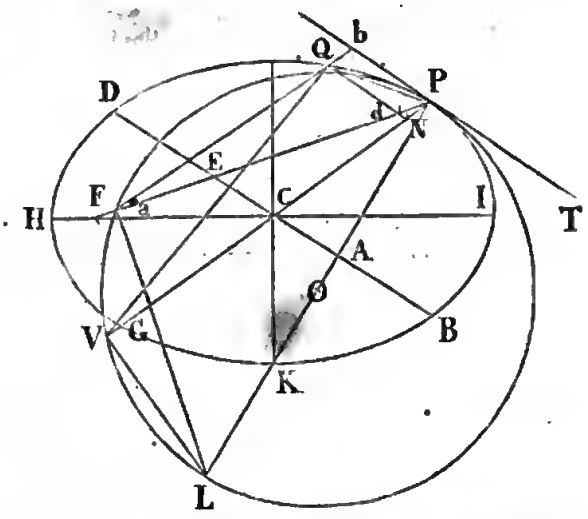
The ordinate BH will at last be equal to the vanishing arc ACB, because if the parallelogram AHBD be completed, BH will always be equal to AD.

COROLLARY 2.

Hence from the common abscissa AH, we have to find the *chord AB* from the nature of the circle; and the *ordinate* HB, from the nature of the curve, and put their values equal to each other.

PROP. I.

To determine the chord of the circle of curvature at any point of an ellipse that passes through the center thereof.



Let IQHK represent an ellipse, C the center, and F the

focus; and let P Q V L be the circle of curvature touching it in P, whose diameter is P L, and P V the chord of curvature passing through the center C. The other lines drawn as per figure.

From the similar triangles P N Q, P V Q, we have P N : P Q :: P Q : P V $= \frac{PQ^2}{PN} = \frac{arc^2\ PQ}{PN}$. $\therefore$ $PQ^2 = PN \times PV$. And (Prop. XVI. ellip.) $PN \times NG : NQ^2 :: PC^2 : CD^2$ $\therefore$ $NQ^2 = \frac{PN \times NG \times CD^2}{PC^2}$. But per lemma 2, $PQ^2 = NQ^2$ $\therefore$ $PN \times PV = \frac{PN \times NG \times CD^2}{PC^2}$, ther. $PV = \frac{NG \times CD^2}{PC^2} = \frac{2PC \times CD^2}{PC^2} = \frac{2CD^2}{PC}$ For when Q moves up to P, N G is equal to 2 P C.

Cor. 1. The chord of curvature P V, passing through the center C, is equal to the parameter of the diameter P G. (Ellip. Def. 18.)

Cor. 2. From the similar triangles P a Q, P d Q, we have P a : P Q :: P Q : P d $\therefore$ $Pa = \frac{QP^2}{Pd}$. Therefore $PN \times PV = Pa \times Pd$.

Prop. II.

To determine the diameter of the circle of curvature at any point of an ellipse.

By the similar triangles P C A, P V L (Fig. to Prop. I.) P A : P C :: P V : P L $= \frac{PV \times PC}{PA}$, but $PV = \frac{2DC^2}{PC}$ (last Prop.) Ther. $PL = \frac{PC}{PA} \times \frac{2DC^2}{PC} = \frac{2DC^2}{PA}$ = diameter of curvature.

Prop. III.

To determine the chord of the circle of curvature at any point of an ellipse that passes through the focus thereof.

Join P F, and let it intersect the circle of curvature in a, and the conjugate diameter D B in E; also join a L. (Fig. to Prop. I.) Then per the similar triangles P A E, P a L; P E : P A :: P L :

$Pa = \frac{PA \times PL}{PE}$, but (Cor. Prop. XXV. ellip.) $PE = CH$, and (Prop. II.) $PL = \frac{2DC^2}{PA}$; ther. $Pa = \frac{PA}{CH} \times \frac{2DC^2}{PA} = \frac{2DC^2}{CH}$ =chord of curvature passing through the focus F.

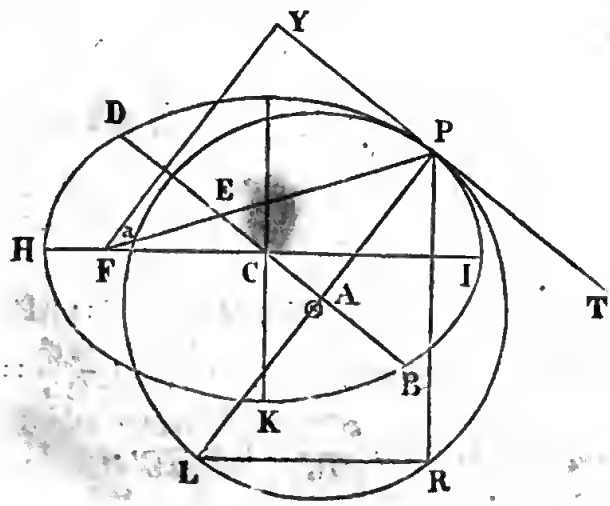
Cor. 1. By Prop. I. $PV = \frac{2CD^2}{PC}$, and by this Prop. $Pa = \frac{2DC2}{CH}$. Hence $PV : Pa :: \frac{1}{PC} : \frac{1}{CH} :: CH : CP$.

Cor. 2. At the extremity K of the minor axis, CD becomes CH, ther. $Pa = \frac{2CD^2}{CD} = 2DC$.

Cor. 3. Since by Prop. I. $PV = \frac{2CD^2}{PC}$, by Prop. II. $PL = \frac{2DC^2}{PA}$, and by Prop. III. $Pa = \frac{2DC^2}{CH}$ we have the diameter of curvature, the chord of curvature passing through the centre, and the chord of curvature passing through the focus respectively, as $\frac{1}{PA}, \frac{1}{PC}, \frac{1}{PE}$.

Prop. IV.

If FY be drawn perpendicular to the tangent at P, and L represent the latus-rectum of the transverse axis, then the diameter of curvature $PL = L \times \text{Sec.}^3 PFY$.



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Per Def. 18. $\frac{1}{2}L = \frac{CK^2}{CH}$, and (Prop. III.) $\frac{1}{2}Pa = \frac{DC^2}{CH}$; ther. $\frac{1}{2}L : \frac{1}{2}Pa :: CK^2 : CD^2$. And per similar triangles, FP: FY :: PE: PA$= \frac{CH \times CK}{CD}$, (Cor. 1. Prop. Ellip. XXXI). Ther. CH : PA :: CD : CK :: FP : FY. Or, $CD^2 : CK^2 :: FP^2 : FY^2 :: Pa : L$.

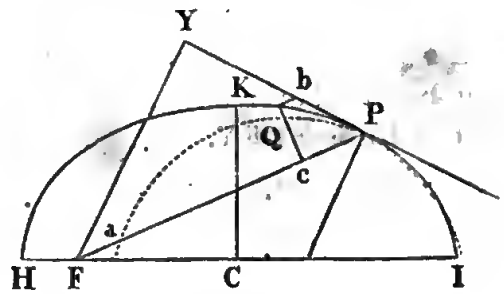
But (Cor. 3. Prop. III.) Pa : PL :: PA : CH :: FY : FP,

And - - - - - - - - L: Pa:: $FY^3 : FP^3 \therefore L : PL :: FY^3 : FP^3$.

Hence $PL = \left(\frac{FP}{FY}\right)^3 \times L = L \times \frac{1}{\cos.^3 PFY} = L \times \sec.^3 PFY$ (rad. = 1).

Prop. V.

If FY be drawn perpendicular to the tangent at P, and the arc PQ be taken indefinitely small, and Qc be perpendicular and Qb parallel to FP; then the latus-rectum of the transverse axis will be equal to $\frac{(Qc)^2}{Qb} = L$.



Since the arc PQ coincides with the tangent at P, the triangle QPc continually approaches towards similarity with the triangle FPY, as Q moves towards P, and at the point of contact (from the similar triangles FPY, PQc) $QP^2 : Qc^2 :: FP^2 : FY^2 ::$ Pa : L (Prop. IV.) divide by Qb, then $\frac{QP^2}{Qb} : \frac{Qc^2}{Qb} :: Pa : L$.

But Cor. 2. Prop. I.) $Pa = \frac{QP^2}{Qb} \therefore L = \frac{Qc^2}{Qb}$.

Prop. VI.

The radius of curvature P⊙ at any point P of an ellipse, is directly as the cube of the semi-diameter C D.

By Cor. 1. Prop. XXXI. Ellip. $PA = \frac{CH \times CK}{CD}$ (Fig. to Prop. IV.)

And (Prop. II.) $P\odot = \frac{DC^2}{PA}$; and by substituting for P A we have $P\odot = DC^2 \times \frac{CD}{CH \times CK} = \frac{CD^3}{CH \times CK}$. But C H and C K are given, therefore the radius of curvature P⊙ is directly as CD^3.

Cor. 1. The radius of curvature at the vertex of the transverse diameter is half the latus-rectum of that diameter.

For when P coincides with I, C D becomes equal to C K, and then $P\odot = \frac{CD^3}{CH \times CK} = \frac{CK^3}{CH \times CK} = \frac{CK^2}{CH}$ = half the latus-rectum of the transverse axis (Def. 18. Ellip.)

Cor. 2. If the point P falls on K, C D will become equal to C H, and then $P\odot = \frac{CD^3}{CH \times CK} = \frac{CH^3}{CH \times CK} = \frac{CH^2}{CK}$ = half the latus-rectum of the conjugate axis (Def. 18. Ellip.).

Cor. 3. The radius of curvature is therefore *least at* I, and *greatest at* K.

Cor. 4. Since the radius of curvature at H, is to the radius of curvature at K, as $CK^3 : CH^3$,

By putting d for the measure of a degree at H,

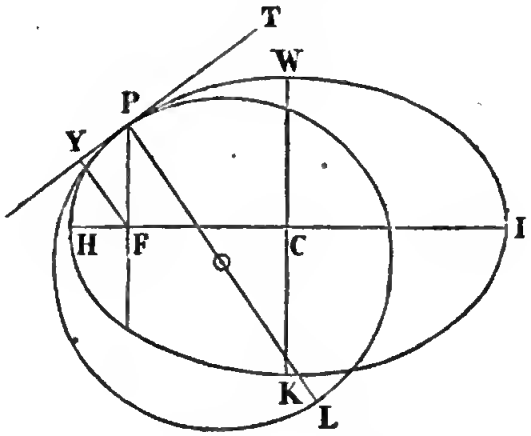
And - - - - D for the measure of a degree at K, we shall have $d : D :: CK^3 : CH^3$ (the lengths of the degrees being as the radii). Hence if d be less than D, C K will be less than C H, which is found to be the case by actual measurements taken at the Equator and near the North Pole of the Earth.

Cor. 5. Any chord of curvature P R, is directly as the sine of the angle R P T, and as the cube of the semi-diameter C D.

For rad. : P L :: sin. P L R : P R; that is, $1 : \frac{2CD^3}{CH \times CK} ::$ sin. R P T : $PR = \frac{2}{CH \times CK} \times CD^3 \times$ sin. R P T.

Prop. VII.

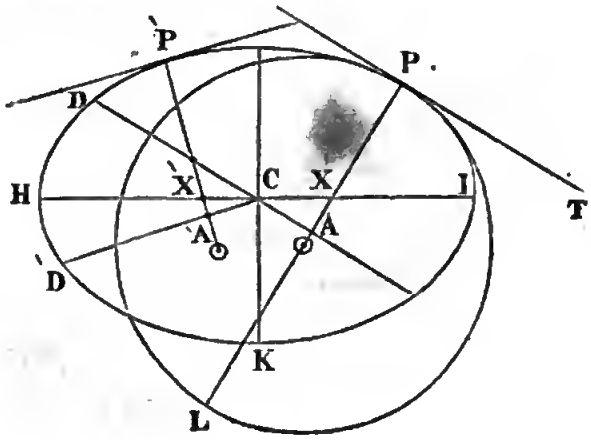
If P T be a focal tangent to the ellipse H W I K, the radius of curvature at the point P is equal to $\frac{FP^4}{FY^3}$.



By Prop. IV. $PL = \frac{FP^3}{FY^3} \times L$, and Def. 6. and Prop. XV. Ellip. $2FP = L$. Ther. $PL = \frac{FP^3}{FY^3} \times 2FP$; or, $\frac{PL}{2} = P\odot = \frac{FP^4}{FY^3}$.

Prop. VIII.

If P A, P X, be perpendicular to the tangent P T of an ellipse, then the radius of curvature P⊙ is *reciprocally* as the cube of P A, or *directly* as the cube of P X.



By Cor. 1. Prop. XXXI. Ellip. $CD^3 = \frac{CH^3 \times CK^3}{PA^3}$, and Prop. VI. $P\odot = \frac{CD^3}{CH \times CK}$, and substituting for CD^3, we have $P\odot = \frac{1}{CH \times CK} \times \frac{CH^3 \times CK^3}{PA^3} = \frac{CH^2 \times CK^2}{PA^3}$, but CH and CK are constant, therefore $P\odot$ is as $\frac{1}{PA^3}$, or $P\odot$ is reciprocally as PA^3.

And since $P\odot \times PA^3 = CH^2 \times CK^2$, and (Prop. XXVII. Ellip.) $PA \times PX = CK^2$; ther. $PA^3 = \frac{CK^6}{PX^3}$; and by substitution $P\odot \times \frac{CK^6}{PX^3} = CH^2 \times CK^2$; or, $P\odot = \frac{CH^2}{CK^4} \times PX^3$; therefore $P\odot$ is directly as PX^3.

Cor. 1. By Prop. XXXVII. Ellip. $PX = \frac{CK^2}{c\sqrt{CH^2 + t^2 \cdot CK^2}}$.

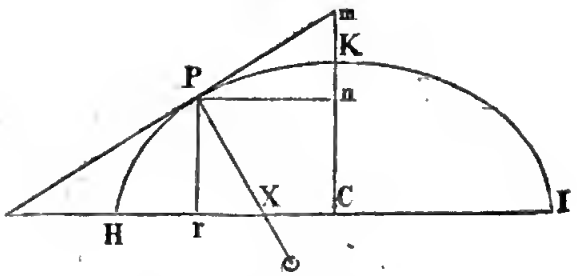
Ther. $\frac{CH^2}{CK^4} \times PX^3 = \frac{CK^6}{(c\sqrt{CH^2 + t^2 \cdot CK^2})^3} \times \frac{CH^2}{CK^4}$.

That is, $P\odot = \frac{CK^2 \times CH^2}{(c\sqrt{CH^2 + t^2 \cdot CK^2})^3}$; c=cos; t=tang. PXI.

Cor. 2. If $P\odot$, the radius of curvature at any other point $\grave{P}$, intersect the transverse axis in $\grave{X}$, and the semidiameter $C\grave{D}$ in $\grave{A}$; and if d represent the measure of a degree at P, and D the measure of a degree at $\grave{P}$, then by the Prop. $P\odot : \grave{P}\odot :: d : D :: \frac{1}{PA^3} : \frac{1}{\grave{P}\grave{A}^3} :: PX^3 : \grave{P}\grave{X}^3$. Or $\frac{D}{d} = \frac{PA^3}{\grave{P}\grave{A}^3} = \frac{\grave{P}\grave{X}^3}{PX^3}$.

Prop. IX.

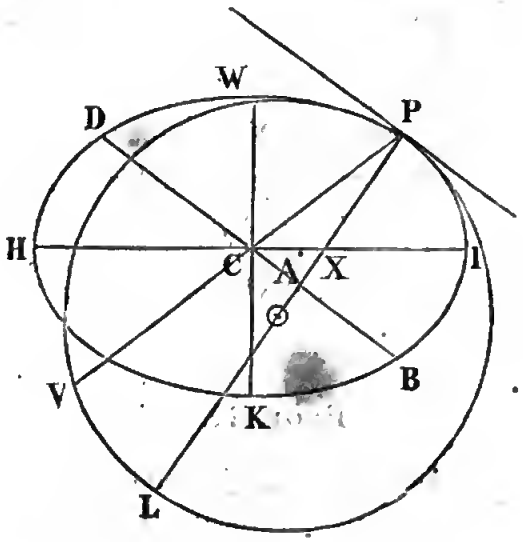
If the radius of curvature $P\odot$, perpendicular to the tangent Pm, intersect the transverse diameter HI in X, and the ordinates Pn, Pr be drawn, then $\frac{\text{Sec.}^3\ n\,P\,m}{P\odot} = \frac{CK^4}{CH^2 \times Cn^2}$.



By Prop. VIII. $P\odot = \frac{C\,H^2}{C\,K^4} \times P\,X^3$. and by trig. rad. : sec. $r\,P\,X :: P\,r : P\,X$; ther. $P\,X = P\,r \times \sec.\, r\,P\,X$, and by substitution we have $P\odot = \frac{C\,H^2}{C\,K^4} \times P\,r^3 \times \sec.^3 r\,P\,X$ $\therefore \frac{\sec.^3 n\,P\,m}{P\odot} = \frac{C\,K^4}{C\,H^2 \times C\,n^3}$.

Prop. X.

If $P\odot$ be the radius of curvature at any point P, B D conjugate to P C, H I and W K the transverse and conjugate axes, and P A perpendicular to B C, then $P\odot$, C B, P A are in continued proportion.



By Prop. LI. $P\odot = \frac{C\,D^3}{CH \times CK}$, and Cor. 1. Prop. XXXI.

Ellip. $PA \times CD = CH \times CK$. Then substituting for $CH \times CK$, we have $P\odot = \frac{CD^3}{PA \times CD} = \frac{CD^2}{PA}$; hence $P\odot$, CD, PA, are in continued proportion.

Prop. XI.

$PA^3 : CB^3 :: \frac{1}{2}$ parameter of the transverse axis HI : rad. curvature $P\odot$. $PA^3 : CK^3 :: \frac{1}{2}$ parameter of the conjugate axis KW : radius of curvature $P\odot$.

By Prop. VIII. $P\odot \times PA^3 = CH^2 \times CK^2 = HC^3 \times \frac{CK^2}{HC}$.

Ther. $PA^3 : CB^3 :: \frac{CK^2}{HC} : P\odot :: \frac{1}{2}$ param. of HI : $P\odot$.

From the same equation $P\odot \times PA^3 = CH^2 \times CK^2 = CK^3 \times \frac{HC^2}{CK}$, we have $PA^3 : CK^3 :: \frac{HC^2}{CK} : P\odot :: \frac{1}{2}$ param. of KW : $P\odot$ (fig. to last Prop.).

Prop. XII.

The chord which the circle of curvature cuts off from a diameter of an ellipse, is a third proportional to that diameter and its conjugate; which chord will be the parameter of the diameter.

By Prop. I. The chord of curvature PV passing through the center C of the ellipse is equal to $\frac{2CD^2}{PC}$; ther. $2PC : 2CD :: 2CD : \frac{2CD^2}{PC} = PV$.

Prop. XIII.

If PX be perpendicular to the tangent at P, intersecting the transverse axis HI in X, and L represent the parameter of the transverse diameter.

Then $\frac{PX^3}{P\odot} = \left(\frac{L}{2}\right)^2$.

By Prop. VIII. $P\odot = \left(\frac{CH}{CK^2}\right)^2 \times PX^3$; but (Def. 18.) $\frac{CK^4}{CH}$

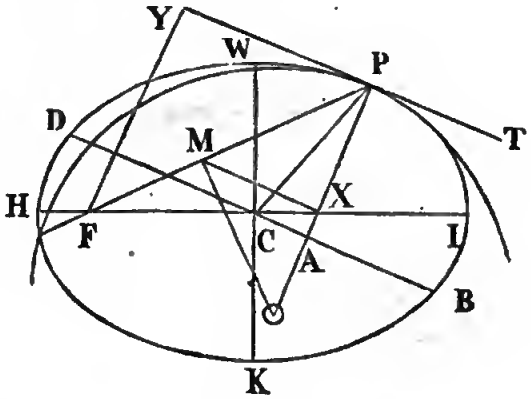
$= \frac{L}{2}$ Ther. $\frac{CH}{CK^2} = \frac{2}{L}$; hence $P\odot = \frac{4 \times PX^3}{L^3} = \left(\frac{2}{L}\right)^3 \times PX^3$.

Ther. $\frac{PX^3}{P\odot} = \left(\frac{L}{2}\right)^2$. (Fig. to last Prop.)

Prop. XIV.

Let a perpendicular be drawn to the tangent at P, intersecting the transverse axis H I in X, and P F drawn to the focus F.

Then if X M be drawn perpendicular to P X, and M ⊙ perpendicular to P F, intersecting P X in ⊙; then will the point ⊙ be the center of the circle of curvature at the point P.



Draw D B conjugate to P C, and demit the perpendicular F Y on the tangent P T.

By Prop. XXVII. Ellip. $PA \times PX = CK^2$

By Prop. II. $PA \times P\odot = CD^2$.

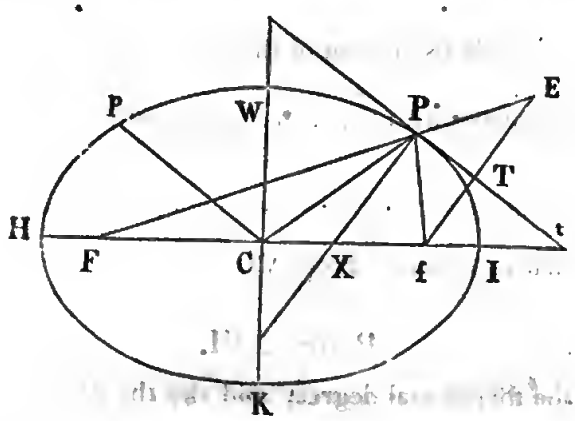
Ther. $CD^2 : CK^2 :: P\odot : PX$.

By Prop. XXXI. Ellip. $FP^2 : FY^2 :: CD^2 : CK^2$, and per. sim. triangles $FP^2 : FY^2 :: P\odot^2 : PM^2 :: P\odot : PX :: P\odot^2 : PX \times P\odot$; that is, $P\odot^2 : PM^2 :: P\odot^2 : PX \times P\odot$ ∴ $P\odot \times PX = PM^2$.

Consequently M X is at right angles to the radius of curvature ⊙P.

Prop. XV.

Having given the axes of an ellipse and the latitude of a point P, to find the radius of curvature in the given latitude.



Let H W I K represent a meridian through the given latitude P; H I, K W the axes, F, f, the foci. Draw P X perpendicular to the tangent P T, and f T E parallel to P X.

Put the semi-transverse C I = a	Sin. of Lat. P X I = sin. E f I = S
semi-conjugate C W = b	Cos. - - - - - - - - - E f I = C
$a^2 - b^2$ - - - - - - = v^2	Radius of curvature at P, = R.

$2\sqrt{a^2-b^2} = F f = 2v$ (Prop. VI.) and (Cor. 2. Prop. XI. Ellip.) F E = H I = 2 a.

Then per trig. F E : sin. F f E :: F f : sin. F E f = sin. T f P (Cor. 1. Prop. XI. Ellip.) That is, $2a : S :: 2v : \frac{V S}{a} =$ sin. T f P; and $1 - \frac{v^2 S^2}{a^2} = \frac{a^2 - v^2 S^2}{a^2} =$ cos.^{2} T f P; and $\frac{(a^2 - v^2 S^2)^{\frac{3}{2}}}{(a^2)^{\frac{3}{2}}}$ $= \frac{(a^2 - v^2 S^2)^{\frac{3}{2}}}{a^3} =$ cos.^{3} T f P (rad. 1.).

Then (Prop. IV.) $R = \frac{b^2}{a} \times \frac{1}{\cos.^3 T f P} = \frac{b^2}{a} \times \frac{a^3}{(a^2 - v^2 S^2)^3} =$

$\frac{a^2 b^2}{(a^2 - v^2 S^2)^{\frac{3}{2}}}$, or, by substituting $1 - C^2$ for S^2; $R = \frac{a^2 b^2}{(b^2 + v^2 C^2)^{\frac{3}{2}}}$.*

When P coincides with I, then $S = O$, and $R = \frac{a^2 b^2}{(a^2)^{\frac{3}{2}}} = \frac{a^2 b^2}{a^3}$

$= \frac{b^2}{a}$ = half the latus-rectum of the transverse axis.

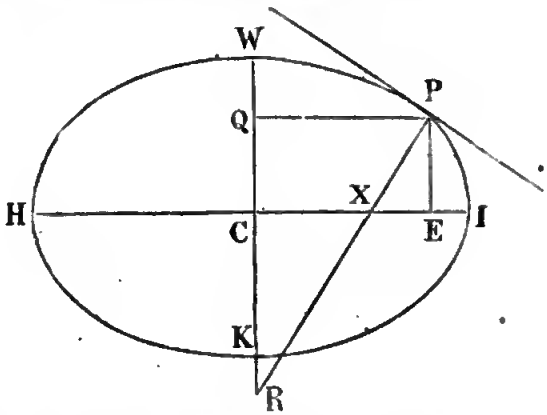
When P coincides with W, then $C = O$, and $R = \frac{a^2 b^2}{(b^2)^{\frac{3}{2}}} = \frac{a^2 b^2}{b^3}$

$= \frac{a^2}{b}$ = half the latus-rectum of the conjugate axis.

See Corollaries 1 and 2. Prop. VI.

Prop. XVI.

Having the meridional degree, and also the degree perpendicular to the meridian, in a given latitude; to find the earth's axes, supposing it an ellipsoid.—(Trigonometrical Survey, vol. i. page 300, and Mathematical Repository, No. 14.)



Suppose I K H W to be the elliptical meridian passing through the point P, whose latitude is given; C I, C W, the equatorial and polar semi-axes. Let P E be the ordinate to the point P,

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* Otherwise thus:

Join C P, and draw its semi-conjugate C p, and we have (Prop. VI.) rad. cur. at H : rad. cur. at P :: $C K^3 : C p^3$; but (Cor. 1. ibid.) The rad.

and draw P R perpendicular to the curve at P, which will be the radius of curvature of the perpendicular degree at that point; also draw P Q parallel to I C.

Put s, c, t for the sine, cosine, and tangent of the given latitude P X E.

P = length of the perpendicular degree.
m = length of the meridional degree.
d = 57°. 29, &c.

Then $C E = Q P = \frac{C I^2}{\sqrt{C I^2 + t^2 C W^2}}$ - - - (Prop. XXXVII. Ellip.)

$P R = \frac{C I^2}{c\sqrt{C I^2 + t^2 C W^2}} = d p$ (Prop. XLIV. Ellip.)

And - - - - - - $\frac{C I^2 \cdot C W^2}{(c\sqrt{C I^2 + t^2 C W^2})^3} = d m$ (Cor. 1. Prop. VIII.)

Therefore $1 : \frac{C W^2}{c^2 \cdot C I^2 + c^2 t^2 \cdot C W^2} :: p : m \therefore \frac{C I^2}{C W^2} = \frac{P}{mc^2} - t^2$.

But $c^2 = \frac{1}{1+t^2} \therefore \frac{C I^2}{C W^2} = \frac{P(1+t^2)}{m} - t^2 = \frac{P + (P-m)t^2}{m}$, which put = V.

Hence $C I^2 = v \times C W^2$; substitute for $C I^2$ in the value of d P,

Then $\frac{v C W^2}{c\sqrt{v C W^2 + t^2 C W^2}} = d P = \frac{v \cdot C W}{c\sqrt{v + t^2}} \therefore C W = \frac{c d P}{v} \sqrt{v + t^2}$.

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cur. at $H = \frac{C K^2}{C H} = \frac{b^2}{a}$; hence $\frac{b^2}{a} : b^1 :: $ rad. cur. at P : $C p^1 \therefore$ rad. cur. at $P = \frac{C p^1}{a b}$. But (Prop. XXXVII. Ellip.) $C p = \frac{a b}{c\sqrt{b^2 + a^2 t^2}} = \frac{a b}{\sqrt{(a^2 - b^2)s^2 + b^2}} = \frac{a b}{\sqrt{b^2 + v^2 s^2}}$, ther. $C p^3 = \frac{a^3 b^3}{(b^2 + v^2 s^2)^{\frac{3}{2}}}$ (s = sin., c = cos., t = tang. of p C H, or of P t H, or of the comp. of P X t, ther. s = C).

The rad. cur. at P being equal to $\frac{1}{ab} \times C p^3$, we have $R = \frac{1}{ab} \times \frac{a^3 b^3}{(b^2 + v^2 C^2)^{\frac{3}{2}}} = \frac{a^2 b^2}{(b^2 + v^2 C^2)^{\frac{3}{2}}}$ as before.

We have $v = \frac{P+(P-m)t^2}{m} = \frac{P+(P-m)\frac{s^2}{c^2}}{m} = \frac{P+\frac{Ps^2-ms^2}{c^2}}{m}$

$= \frac{Pc^2+Ps^2-ms^2}{mc^2} = \frac{P-ms^2}{mc^2}$.

And $v+t^2 = \frac{P-ms^2}{mc^2} + \frac{s^2}{c^2} = \frac{Pc^2-ms^2c^2+ms^2c^2}{mc^4} = \frac{Pc^2}{mc^4}$

$= \frac{P}{mc^2}$.

Ther. $CW = \frac{cdP}{v}\sqrt{v+t^2} = cdP \times \frac{mc^2}{P-ms^2} \times \frac{1}{c}\sqrt{\frac{P}{m}} =$

$\frac{c^2dPm}{P-ms^2}\sqrt{\frac{P}{m}} = \frac{c^2dP\sqrt{Pm}}{P-ms^2} = \frac{c^2d\sqrt{Pm}}{1-\frac{m}{P}s^2} = \frac{c^2d\sqrt{Pm}}{(1+s\sqrt{\frac{m}{P}})(1-s\sqrt{\frac{m}{P}})}$.

$CI^2 = v \times CW^2 = v \times \frac{c^2d^2P^2}{v^2} \times (v+t^2) = \frac{c^2d^2P^2}{v} \times (v+t^2) =$

$c^2d^2P^2 \times \frac{mc^2}{P-ms^2} \times \frac{P}{mc^2} = \frac{c^2d^2P^2}{P-ms^2} = \frac{c^2d^2P^2}{1-\frac{m}{P}s^2}$.

Ther. $CI = \frac{cdP}{\sqrt{1-\frac{m}{P}s^2}} = \frac{cdP}{\left\{(1+s\sqrt{\frac{m}{P}})(1-s\sqrt{\frac{m}{P}})\right\}^{\frac{1}{2}}}$.

Corollary 1.

If l = the length of a *degree* of *longitude* at the point P, then PQ will be its radius of curvature and equal to dl, and $PR = dp$, therefore $PQ : PR :: 1 ; P :: c : 1 \therefore P = \frac{1}{c}$, which being substituted in the above expressions, we have

$CW = \frac{c^2d\sqrt{Pm}}{1-\frac{m}{P}s^2} = \frac{dc\sqrt{cml}}{1-\frac{mc}{l}s^2} = \frac{dc\sqrt{cml}}{(1+s\sqrt{\frac{mc}{l}})(1-s\sqrt{\frac{mc}{l}})}$, and

$CI = \frac{cdP}{\sqrt{1-\frac{m}{P}s^2}} = \frac{dl}{\left\{(1+s\sqrt{\frac{mc}{l}})(1-s\sqrt{\frac{mc}{l}})\right\}^{\frac{1}{2}}}$

COROLLARY 2.

Because $\frac{C I^2}{c\sqrt{C I^2 + t^2 C W^2}} = P R = d P$; if b and T represent the *cosine* and *tangent* of some other latitude, and P the *perpendicular degree* in that latitude; then $\frac{C I^2}{b\sqrt{C I^2 + T^2 C W^2}} = d P$, the radius of curvature where the perpendicular P is measured. Hence $\frac{C I^2}{P b\sqrt{C I^2 + T^2 C W^2}} = d$, and the former equation gives $\frac{C I^2}{P c\sqrt{C I^2 + t^2 C W^2}} = d$; these being equated, we have $P^2 b^2 (C I^2 + T^2 C W^2) = P^2 c^2 (C I^2 + t^2 C W^2)$.

Let s and S be the *sines* to the *cosines* c and b; put $\frac{s^2}{c^2}$ and $\frac{S^2}{b^2}$ for t^2 and T^2, and we shall have $P^2 b^2 (C I^2 + \frac{S^2}{b^2} \cdot C W^2) = P^2 c^2 (C I^2 + \frac{s^2}{c^2} \cdot C W^2$.

And by multiplying and transposing, we get $(P^2 b^2 - p^2 c^2) \times C I^2 = (p^2 s^2 - P^2 S^2) \times C W^2$; hence we have $C I : C W :: \sqrt{p^2 s^2 \backsim P^2 S^2} : \sqrt{P^2 b^2 \backsim p^2 c^2}$, for the ratio of the axes; which being expounded by 1 : n, we have $C I^2 : C W^2 :: n^2 : 1$ $\therefore C I^2 = n^2 \times C W^2$; this substituted for $C I^2$ in the equation $\frac{C I^2}{p c\sqrt{C I^2 + t^2 C W^2}}, = d$, gives $\frac{n^2 \cdot C W^2}{p c\sqrt{n^2 \cdot C W^2 + t^2 \cdot C W^2}} = \frac{n^2 C W}{p c\sqrt{n^2 + t^2}} = d \therefore C W = \frac{d c P}{n^2}\sqrt{n^2 + t^2}$. Whence $C I = n \times C W^2 = \frac{d c P}{n}\sqrt{n^2 + t^2}$. $n = \sqrt{\frac{p^2 s^2 \backsim P^2 S^2}{P^2 b^2 \backsim p^2 c^2}}$.

Hence if l and L be the *degrees* of *longitude* in the given latitudes, we have $p = \frac{l}{c}$, and $P = \frac{L}{b}$, which substituted for p and P in the proportion $C I : C W :: \sqrt{p^2 s^2 \backsim P^2 S^2} : \sqrt{P^2 b^2 \backsim p^2 c^2}$, we shall have $C I^2 : C W^2 :: l^2 t^2 \backsim L^2 T^2 : L^2 \backsim l^2 :: \pi^2 : 1$.

Ther. $\pi = \sqrt{\frac{l^2 t^2 \backsim L^2 T^2}{L^2 \backsim l^2}}$; hence $CW = \frac{dl}{\pi^2}\sqrt{\pi^2 + t^2}$, and $CI = \frac{dl}{\pi}\sqrt{\pi^2 + t^2}$.

If $Cl^2 = n^2 \times CW^2$; or $CI^2 = \pi^2 \times CW^2$ be substituted in the equation $\frac{Cl^2}{Pb\sqrt{Cl^2 + T^2 \cdot CW^2}} = d$, we shall obtain

$$CW = \frac{dbP}{n^2}\sqrt{n^2 + T^2}, \text{ and } CI = \frac{dbP}{n}\sqrt{n^2 + T^2}.$$

Or,

$$CW = \frac{dl}{\pi^2}\sqrt{\pi^2 + T^2}, \text{ and } CI = \frac{dl}{n}\sqrt{\pi^2 + T^2}.$$

The values of n and π remaining the same.

Corollary 3.

If H = an arc, corresponding to the nat. sin. $s\sqrt{\frac{m}{p}}$ (rad. 1.) then by trigonometry, $(1 + s\sqrt{\frac{m}{p}})(1 - s\sqrt{\frac{m}{p}}) =$ sin. $(90° + H)$ sin. $(90° - H) =$ cos.^{2} H. And by substitution $CW = d\sqrt{pm} \times \frac{\text{cos.}^2 \text{ lat.}}{\text{cos.}^2 H}$, $CI = dp \times \frac{\text{cos. lat.}}{\text{cos. H}}$, and $\frac{CW}{CI} = \sqrt{\frac{m}{p}} \times \frac{\text{cos. lat.}}{\text{cos. H}}$.

If $s\sqrt{\frac{m}{p}}$ be considered as a cosine, then the *sine* of H must be used.

Corollary 4.

If G = an arc, corresponding to the nat. sin. $s\sqrt{\frac{mc}{l}}$, then $(1 + s\sqrt{\frac{mc}{l}})(1 - s\sqrt{\frac{mc}{l}}) =$ sin. $(90° + G)$ sin. $(90° - G) =$ cos.^{2} G. And by substitution $\frac{dc\sqrt{cml}}{\text{cos.}^2 G} = CW$, $\frac{dl}{\text{cos. G}} = CI$, and $\frac{CW}{CI} = \frac{l \times \text{cos. G}}{c\sqrt{cml}}$.

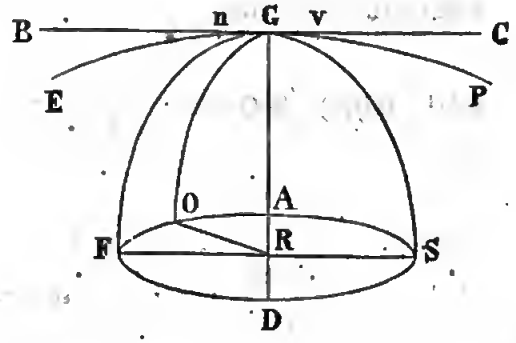
If $s\sqrt{\frac{mc}{l}}$ be considered as a cosine, then the *sine* of G must be used.

In a Sphere, whose Radius is Unity.

$p = P$, $m = p$, $\frac{b}{c} = \frac{L}{l}$, $m c = l$, and $d m = l$; therefore C I = C W.

Prop. XVII.

Having the length of a degree in the meridian, and also that of a degree perpendicular to it, at the same point, to find the length of a degree in any other given direction, supposing the earth to be an ellipsoid. (By Mr. Dalby, in the Philosophical Transactions for 1803, and Leybourn's Diary.)



Let E P represent part of the elliptic meridian, and n v the measure of a degree thereon; bisect n v in G, and draw B C a tangent at G; draw G R at right angles to B C, and equal to the radius of curvature belonging to the degree perpendicular to the meridian, and through R draw F S at right angles to G R.

Conceive another ellipsoid F G S O (whose semi-revolving axe is G R, and fixed axe F S) to touch the given one in the point G, so that the convex part of the one shall coincide with the concave part of the other in that point; let F A S D be a section of F G S O through the center R, perpendicular to R G, and let O G R be the plane of an ellipse, inclined to the meridian E G P, or to the plane F G S, in a given angle F R O.

Let d = 57·29, &c. the degrees in a circular arc, equal to radius,

m = length of the meridional degree,

d m = radius of curvature corresponding to m,

p = length of the perpendicular degree,

$dp = RG = RA$ = radius of curvature corresponding to p,

s = sine, and c = cos. of the angle FRO.

Then in the elliptic quadrant FRG (FRA) are given the semi-transverse axis $RG = dp$, and the radius of curvature at G, in direction EGP, or $FGS = dm$, to find the semi-conjugate axis RF.

(Cor. I. Prop. VI.) $\frac{FR^2}{RG} = dm \therefore FR^2 = Rg \times dm = p\,m\,d^2$.

In the elliptic quadrant FRA are given $FR^2 = p\,m\,d^2$; $RA = dp$; and s and c, the sine and cos. of the angle FRO, to find RO^2.

(Prop. XXXVII. Ellip.) $RO^2 = \frac{RF^2 \times RA^2}{s^2\,RF^2 + c^2\,RA^2} = \frac{m\,p^3\,d^4}{p\,m\,d^2\,s^2 + p^2\,d^2\,c^2} = \frac{m\,p^2\,d^2}{m\,s^2 + p\,c^2}$.

In the elliptic quadrant GRO (right angled at R), are given $GR = dp$, and $RO^2 = \frac{m\,p^2\,d^2}{m\,s^2 + p\,c^2}$; to find R the radius of curvature at the vertex G, in Direction GO. (Cor. Prop. VI.) $R = \frac{RO^2}{RG} = \frac{m\,p^2\,d^2}{m\,s^2 + p\,c^2} \div dp = \frac{m\,p\,d}{m\,s^2 + p\,c^2}$, therefore $\frac{R}{d} = \frac{m\,p}{m\,s^2 + p\,c^2}$ $= \frac{m\,p}{m\,s^2 - p\,s^2 + p} = \frac{m\,p}{p - (p-m)s^2}$ = length of a degree oblique to the meridian. $(1 - s^2 = c^2)$.

Cor. 1. If Δ and D represent the lengths of two oblique degrees; s, c, and S, C, the respective sines and cosines of their inclination to the meridian; then $\Delta = \frac{p\,m}{m\,s^2 + p\,c^2}$, and $D = \frac{p\,m}{m\,S^2 + p\,C^2}$; from whence $p = \frac{\Delta\,m\,s^2}{m - \Delta\,c^2} = \frac{D\,s^2\,m}{m - C^2\,D}$, and $m = \frac{\Delta\,c^2\,p}{p - s^2\,\Delta} = \frac{C^2\,D\,p}{p - S^2\,D}$; therefore $m = \frac{S^2\,c^2 - C^2\,s^2}{S^2\,D - s^2\,\Delta} \times \Delta\,D$, and $p = \frac{S^2\,c^2 - C^2\,s^2}{c^2\,\Delta - C^2\,D} \times \Delta\,D$, the meridional and perpendicular degrees, exhibited in terms of the oblique degrees, combined with

the sines and cosines of their inclinations to the meridian. Therefore an ellipsoid may be determined from the lengths of two oblique degrees in the same latitude.

We may remark, that $\frac{p\,m}{p-(p-m)s^2}$ will give the oblique degree on different spheroids, because the expression is in terms of the meridional and perpendicular degrees and the sine of the obliquity only.

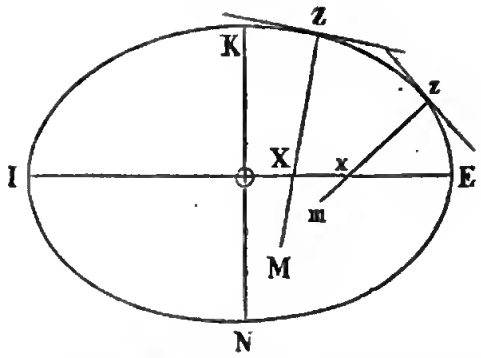
When R O coincides with R A, then s=1 and c=o; in that case $\Delta = \frac{p\,m}{m\,s^2+p\,c^2} = \frac{p\,m}{m} = p$.

Cor. 2. Since $\Delta = \frac{p\,m}{p-(p-m)s^2} = \frac{m}{1-\frac{p}{p-m}s^2} = \frac{m}{1-v^2s^2} = \frac{m}{(1+v\,s)(1-v\,s)}$ by putting $\frac{p}{p-m} = v^2$. If A = an arc corresponding to the nat. sin. v s (radius unity), then per trig. $(1+v\,s)(1-v\,s) = \sin.(90°+A)\ \sin.(90°-A) = \cos.^2 A$; therefore $\Delta = \frac{m}{\cos.^2 A}$. Hence the following rule.

Find the log. $s\sqrt{\frac{p-m}{p}}$ among the log. sines;
Double the log. cosine corresponding thereto,
Subtracted from the logarithm of m, and
The remainder will be the logarithm of the oblique degree.

Prop. XVIII.

To find the earth's axes, having given the measures of a degree of a meridian in two given latitudes, the earth being supposed an ellipsoid.—(*Mathematical Repository, No.* 15, *November,* 1815.)



Let I K E N represent a meridian of the earth, K N the less, and I E the greater axis, and z, z, the given latitudes; at right angles to which let Z X, z x be drawn, and let Z M, z m represent the radii of curvature at Z and z.

Put A=degrees in the greater latitude; S=sine, C=cosine;
B=degrees in the less latitude; s=sine, c=cosine; rad. 1.
D=length of a degree on the meridian, in latitude A,
d =length of a degree on the meridian, in latitude B,
E =an arc, corresponding to the nat. sin. $s\sqrt[3]{\frac{d}{D}} = v\,s$,
F =an arc, corresponding to the nat. cosine $c\sqrt[3]{\frac{d}{D}} = v\,c$,
57·29, &c. × D=R=Z M; 57·29, &c. × d=r=z m,
57·29, &c. being the degrees in a circular arc equal to rad.
Put also $O\,E^2=a^2$, $O\,K^2=b^2$, and L=principal latus-rectum.

Then Prop. XV. $\left(\frac{a^2-(a^2-h^2)\,S^2}{a^2-(a^2-b^2)\,s^2}\right)^{\frac{3}{2}} = \frac{d}{D}$ $\therefore (1-v^2)\,a^2 = (S^2-v^2\,s^2)\,(a^2-b^2)$, and thence $b^2 : a^2 :: v^2\,c^2-C^2 : S^2-v^2\,s^2$;

Or, $\frac{O\,K^2}{O\,E^2}$* $= \frac{(v\,c+C)\,(v\,c-C)}{(S+v\,s)\,(S-v\,s)}$. Hence the ratio of O K to O E is known.

The proportion of the diameters being known, the principal latus-rectum may be found as follows.

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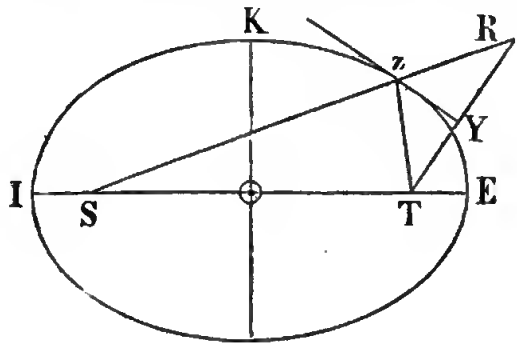
* $\frac{O\,K^2}{O\,E^2} = \frac{(v\,c+C)\,(v\,c-C)}{(S+v\,s)\,(S-v\,s)} = \frac{(c\,d^{\frac{1}{3}}+C\,D^{\frac{1}{3}})\,(c\,d^{\frac{1}{3}}-C\,D^{\frac{1}{3}})}{(S\,D^{\frac{1}{3}}+s\,d^{\frac{1}{3}})\,(S\,D^{\frac{1}{3}}-s\,d^{\frac{1}{3}})}$, by substituting for v. The same as given in vol. iii. page 144, Doct. Hutton's Course.

From the foci S, T, draw S z, T z; also draw T R at right angles to the tangent z y, meeting S z produced in R.



Then (6. Ellip.) $\frac{S\,T}{2} = \sqrt{(a^2-b^2)}$, and (Prop. IV.) $\frac{L}{2} = \frac{O\,K^2}{O\,E}$ $= \cos.^3 Y\,T\,z \times r$. And per trig. $a : \sqrt{(a^2-b^2)} :: \sqrt{(S^2-v^2 s^2)} : \sqrt{(1-v^2)} :: s : \sin. Y\,T\,z = s\sqrt{\frac{1-v^2}{S^2-v^2 s^2}}$, hence the $\cos. Y\,T\,z$ $= \sqrt{\frac{S^2-s^2}{S^2-v^2 s^2}}$; therefore $\frac{O\,K^2}{O\,E} = \left(\frac{(S+s)\,(S-s)}{(S+v\,s)\,(S-v\,s)}\right)^{\frac{3}{2}} \times r.$

By using the sine (S) of the greater latitude, we have $\frac{O\,K^2}{O\,E} =$ $\left(\frac{(S+s)\,(S-s)}{(S+v\,s)\,(S-v\,s)}\right)^{\frac{3}{2}} \times v^3 \times R.$

Hence by having the values of $\frac{O\,K^2}{O\,E^2}$ and $\frac{O\,K^2}{O\,E}$ as above, we readily obtain

$$O\,K = \left(\frac{(S+s)\,(S-s)}{(v\,c+C)\,(v\,c-C)}\right)^{\frac{1}{2}} \times \frac{(S+s)\,(S-s)}{(S+v\,s)\,(S-v\,s)} \times r,$$

$$O\,E = \left(\frac{(S+s)\,(S-s)}{(S+v\,s)\,(S-v\,s)}\right)^{\frac{1}{2}} \times \frac{(S+s)\,(S-s)}{(v\,c+C)\,(v\,c-C)} \times r.$$

But by Trigonometry,

$(S+s)\,(S-s) = \sin.(A+B) \sin.(A-B)$,

$(S+v\,s)\,(S-v\,s) = \sin.(A+E) \sin.(A-E)$,

$(v\,c+C)\,(v\,c-C) = \sin.(A+F) \sin.(A-F)$. Therefore by substitution,

$$\frac{OK^2}{OE^2} = \frac{(vc+C)(vc-C)}{(S+vs)(S-vs)} = \frac{\sin.(A+F)\sin.(A-F)}{\sin.(A+E)\sin.(A-E)},$$

$$\frac{OK^2}{OE} = \left(\frac{(S+s)(S-s)}{(S+vs)(S-vs)}\right)^{\frac{3}{2}} \times r = \left(\frac{\sin.(A+B)\sin.(A-B)}{\sin.(A+E)\sin.(A-E)}\right)^{\frac{3}{2}}$$

$\times 57{\cdot}29$ &c. $\times$ d. And thence

$$OE = \left(\frac{S.(A+B)\ S.(A-B)}{S.(A+E)\ S.(A-E)}\right)^{\frac{1}{2}} \times \frac{S.(A+B)\ S.(A-B)}{S.(A+F)\ S.(A-F)} \times 57{\cdot}29$$

&c. $\times$ d,

$$OK = \left(\frac{S.(A+B)\ S.(A-B)}{S.(A+F)\ S.(A-F)}\right)^{\frac{1}{2}} \times \frac{S.(A+B)\ S.(A-B)}{S.(A+E)\ S.(A-E)} \times 57{\cdot}29$$

&c. $\times$ d,

$$\frac{OK}{OE} = \left(\frac{S.(A+F)\ S.(A-F)}{S.(A+E)\ S.(A-E)}\right)^{\frac{1}{2}} \text{ - - - - - - - - - - - - (A).}$$

If the earth be a sphere, then D=d, and E=F=B; hence

$OE=OK=57{\cdot}29$, &c. $\times$ d - - - - - - - - - - - - - - - - (B).

If one of the places as B have no latitude, then B and E will be nothing, and F an arc corresponding to the cos. $\sqrt[3]{\frac{d}{D}}$, the arc will not exceed six degrees, therefore less than A.

Then $OE = \frac{\sin.^2 A}{\sin.(A+F)\sin.(A-F)} \times 57{\cdot}29$, &c. $\times$ d,

$$OK = \frac{\sin. A}{\left(\sin.(A+F)\sin.(A-F)\right)^{\frac{1}{2}}} \times 57{\cdot}29, \text{ \&c. } \times \text{ d},$$

$$\frac{OK}{OE} = \frac{\left(\sin.(A+F)\sin.(A-F)\right)^{\frac{1}{2}}}{\sin. A} \text{ - - - - - - - - (C).}$$

$$\frac{OK}{OE} = \frac{\left\{\sin.(45^\circ+F)\cos.(45^\circ+F)\right\}^{\frac{1}{2}}}{\sin. 45^\circ} \text{ - - - - - (D).}$$

If $A = 45^\circ$, and B less than A. Then

$$OE = \left(\frac{\sin.(45^\circ+B)\cos.(45^\circ+B)}{\sin.(45^\circ+E)\cos.(45^\circ+E)}\right)^{\frac{1}{2}} \times \frac{\sin.(45^\circ+B)\cos.(45^\circ+B)}{\sin.(45^\circ+F)\cos.(45^\circ+F)}$$

$\times 57{\cdot}29$, &c. $\times$ d.

$$OK=\left(\frac{\sin.(45°+B)\cos.(45°+B)}{\sin.(45°+F)\cos.(45°+F)}\right)^{\frac{1}{2}}\times\frac{\sin.(45°+B)\cos.(45°+B)}{\sin.(45°+E)\cos.(45°+E)}$$
$$\times 57{\cdot}29, \&c. \times d,$$

$$\frac{OK}{OE}=\left(\frac{\sin.(A+F)\cos.(A+F)}{\sin.(45°+E)\cos.(45°+E)}\right)^{\frac{1}{2}} \quad \text{- - - - - - - - - - (E).}$$

If $A=45°$, and $B=0$.

Then $OE=\dfrac{\sin.^2 45°}{\sin.(45°+F)\cos.(45°+F)}\times 57{\cdot}29, \&c. \times d,$

$$OK=\frac{\sin.45°}{\left\{\sin.(45°+F)\cos.(45°+F)\right\}^{\frac{1}{2}}}\times 57{\cdot}29, \&c.\times d,$$

Cor. 1. If $s=0$, then $c=1$; we then have $OK=\dfrac{rS}{\sqrt{v^2-C^2}}$, $OE=\dfrac{rS^2}{v^2-C^2}$, and $\dfrac{OK^2}{OE^2}=\dfrac{v^2-C^2}{S^2}=\dfrac{(v+C)(v-C)}{S^2}$.

Cor. 2. Since $OE : OK :: S : \sqrt{v^2-C^2}$, we then get $\dfrac{OE-OK}{OE}=1-\dfrac{\sqrt{v^2-C^2}}{S}=1-\dfrac{\sqrt{(v+C)(v-C)}}{S}$.

Cor. 3. Because $OE^2 : OK^2 :: S^2 : v^2-C^2$, and $OE^2 : OE^2-OK^2 :: S^2 : S^2-(v-C^2) :: S^2 : 1-v^2$.

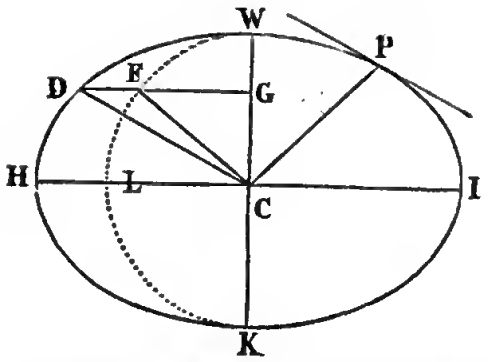
Ther. $\dfrac{OE^2-OK^2}{OE^2}=\dfrac{1-v^2}{S^2}=\dfrac{(OE+OK)(OE-OK)}{OE^2}$,

Or, $\dfrac{2OE(OE-OK)}{OE^2}=\dfrac{2(OE-OK)}{OE}=\dfrac{1-v^2}{S^2}$, nearly.

Ther. $\dfrac{OE-OK}{OE}=\dfrac{(1+v)(1-v)}{2S^2}$, nearly.

Prop. XIX.

To find the earth's axes, by having two degrees of a meridian measured, one at the equator, and the other in a given latitude. —*(Robertson's Navigation.)*



Let H I be the transverse, and K W the conjugate axis; P the given place on the meridian I W H K, and C D conjugate to C P, forming the angle D C W, equal to the latitude of P; on K W describe the circle W F L K, and draw D G parallel to H I, intersecting the circle in F, and join C F.

Put M, for the measure of a degree at the equator,
N, the measure of a degree at P,
A, to represent the angle G C F,
T, the nat. tang. lat. D C W, (radius unity),
t, the nat. tang. of A,
p = 57·29, &c. the degrees in the circular arc, equal to radius.

Then per trig. C D : C F :: sin. G F C : sin. G D C :: cos. G C F : cos. G C D.

But (Prop. VI.) C D : C F (C W) :: $\sqrt[3]{N} \cdot \sqrt[3]{M}$ :: cos. A : cos. lat.

Ther. cos. lat. $\times \sqrt[3]{N} =$ cos. A $\times \sqrt[3]{M}$ $\therefore$ cos. A $= \left(\frac{N}{M}\right)^{\frac{1}{3}} \times$ cos. lat.

Hence the angle A is known.

Again (7. Ellip.) G D : G F :: tang. G C D : tang. G C F :: T : t :: C H : C L :: C H : C W $\therefore \frac{C W}{C H} = \frac{t}{T}$, a given ratio. (Cor. 1. Prop. VI.) $p \times M = \frac{C W}{C H} \times C W = \frac{t}{T} \times C W \therefore C W = \frac{T}{t} \times pM$.

But $T \times C W = t \times C H \therefore C H = \frac{T}{t} \times C W = \frac{T^2}{t^2} \times pM$.

That is, $CW = \frac{\text{tang. lat.}}{\text{tang. A.}} \times pM$, and $CH = \left(\frac{\text{tang. lat.}}{\text{tang. A}}\right)^2 \times pM$.

Prop. XX.

If D and d represent the lengths of a meridional degree in two given latitudes, the sines of which being S and s respectively, and $CK : CH :: 1 : 1+e$ (fig. Prop. XIX.) to determine an approximate value of e, and thence the ratio of the earth's axes.—(*Vince's Astronomy*, vol. ii.)

Let $CH = a$, and $CK = b$; then (Prop. XV.) $\frac{d}{D} = \left(\frac{a^2-(a^2-b^2)S^2}{a^2-(a^2-b^2)s^2}\right)^{\frac{3}{2}}$; and $a^2 : b^2 :: (1+e)^2 : 1$; or, $a^2-b^2 : a^2 :: (1+e)^2-1 : (1+e)^2 :: 2e : 1+2e :: 1 : \frac{1}{2e}+1 :: 1 : \frac{1}{2e}$. Ther. $(a^2-b^2) \times \frac{1}{2e} = a^2$; that is, $a^2-b^2 = 2e a^2$ (omitting e^2, and considering $\frac{1}{2e}+1 = \frac{1}{2e}$). Hence by substitution, $\frac{d}{D} = \left(\frac{a^2-2e a^2 S^2}{a^2-2e a^2 s^2}\right)^{\frac{3}{2}} = \left(\frac{1-2e S^2}{1-2e s^2}\right)^{\frac{3}{2}} = \left(\frac{1-e S^2}{1-e s^2}\right)^3 = \frac{1-3e S^2}{1-3e s^2}$, nearly. From which equation $e = \frac{D-d}{3(S^2 D-s^2 d)}$.

But $CK : CH :: 1 : 1+e :: \frac{1}{e} : \frac{1}{e}+1$; that is, $CK : CH :: \frac{3S^2D-3s^2d}{D-d} : \frac{3S^2D-3s^2d}{D-d}+1$ nearly the required ratio (a).

If $CH : CK :: 1 : m :: 1+e : 1$ (by the preceding). Then $m \times (1+e) = 1 \therefore me = 1-m$; or $1 : m :: e : 1-m$. But, on account of CH being nearly equal to CK, we may consider m nearly equal to unity; and therefore $e = 1-m = \frac{D-d}{3(S^2D-s^2d)}$ $\therefore m = 1 - \frac{D-d}{3S^2D-3s^2d} = 1 - \frac{r}{w}$.

But $CH : CK :: 1 : m :: 1 : 1-\frac{r}{w} :: w : w-r :: \frac{w}{r} : \frac{w}{r}-1$.

Ther. $CH : CK :: \frac{3DS^2-3s^2d}{D-d} : \frac{3S^2D-3s^2d}{D-d} - 1$, nearly (b).

Cor. 1. If d be the measure of a meridional degree at the equator, then s=0.

And (a) $CK : CH :: \frac{3S^2D}{D-d} : \frac{3S^2D}{D-d} + 1$.

Cor. 2. By supposing d and s, as in corollary 1.

Then (b) $CH : CK :: \frac{3S^2D}{D-d} : \frac{3S^2D}{D-d} - 1$.

Cor. 3. If s=0, then by the proposition $D : d :: 1 : 1-3S^2e$, from whence $D = \frac{d}{1-3S^2e} = d(1+3S^2e)$ nearly.

Cor. 4. Since $D=d+3deS^2$; or $D-d=3deS^2$, we shall have D−d as S^2, because 3de is given; that is, the degrees of the meridian from the equator towards the poles are increased in a duplicate ratio of the sine of latitude nearly.

Prop. XXI.

To render the expression $\frac{3S^2D-3s^2d}{D-d}$, as determined by Prop. XX., applicable to the use of logarithms. $\frac{3S^2D-3s^2d}{D-d} = (DS^2-ds^2) \times \frac{3}{D-d} = (1-\frac{ds^2}{DS^2}) \times \frac{3DS^2}{D-d} = \left\{(1+\frac{s}{S}\sqrt{\frac{d}{D}})(1-\frac{s}{S}\sqrt{\frac{d}{D}})\right\} \times \frac{3DS^2}{D-d} = \text{sin. } (90°+A) \text{ sin. } (90°-A) \times \frac{3DS^2}{D-d} = \frac{\text{Cos.}^2 A \times S^2 \times 3D}{D-d}$. A = an arc corresponding to the sin. $\frac{s}{S}\sqrt{\frac{d}{D}}$; an expression very convenient for logarithms.

Corollary. The expression in Cor. 3. Prop. XX. may be reduced as follows:

$$D = \frac{d}{1-3eS^2} = \frac{d}{(1+S\sqrt{3e})(1-S\sqrt{3e})} = \frac{d}{\cos.^2 B}.$$

B= an arc corresponding to the sin $S\sqrt{3e}$.

Prop. XXII.

In a given ellipse, to find an approximate value for the radius of curvature in a given latitude.

Put a = semi-transverse diameter,
b = semi-conjugate, $a - b = c$,
S = sine of latitude, and
R = radius of curvature in the given latitude.

Then Prop. XV. $R = \dfrac{a^2 b^2}{\left(a^2 - (a^2 - b^2) S^2\right)^{\frac{3}{2}}}$.

But $b^2 = a^2 - 2ac + c^2 = a^2 - 2ac$ $\therefore a^2 - b^2 = 2ac$ (omitting c^2).

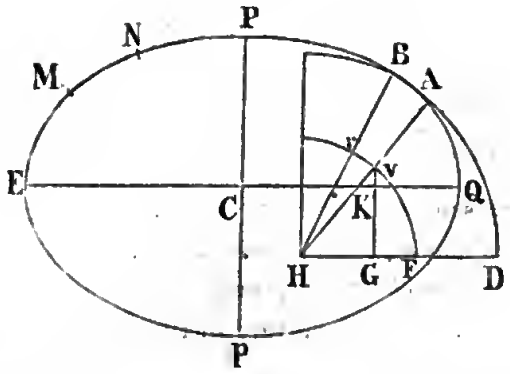
Then by substitution $R = \dfrac{a^2 b^2}{\left(a^2 - 2ac\,S^2\right)^{\frac{3}{2}}} = \dfrac{a^2 b^2}{(a - c\,S^2)^3} = \dfrac{a^2 b^2}{a^3 - 3a^2 c\,S^2} = \dfrac{b^2}{a - 3c\,S^2} = \dfrac{a^2 - 2ac}{a - 3c\,S^2} = \dfrac{a - 2c}{1 - \dfrac{3c}{a} S^2} = a - 2c + 3c\,S^2$, nearly, omitting c^2 &c. which is P. Frisi's Rule. See *Gentleman's Diary*, 1806.

Cor. $1 - \dfrac{3c}{a} S^2 = \left(1 + S\sqrt{\dfrac{3c}{a}}\right)\left(1 - S\sqrt{\dfrac{3c}{a}}\right)$; find an arc A, corresponding to the sin. $S\sqrt{\dfrac{3c}{a}}$; then $R = \dfrac{a - 2c}{\cos.^2 A}$.

Prop. XXIII.

To determine the ratio of the earth's axes from the measures of convenient portions of a meridian in any two given latitudes; the earth being supposed a spheroid generated by the rotation of an ellipse upon its minor axis. *Doct. Gregory's Trig.*

Let the ellipse P E p Q represent a terrestrial meridian passing through the poles P, p, and cutting the equator in E, Q; C the center, C Q the radius of the equator, and P C half the polar axis.



Let A B be an indefinitely small arc of the meridian, whose radius of curvature is A H; join H B; draw H D parallel to E Q; on the center H, and with the radius H F = 1, describe the circular arc F v r, and draw the sine v G; also on the center H, and with the radius of curvature H A, describe the circular arc D A B.

Let C Q=a, C P=b, a−b=c, S=v G=sin. lat.=sin. angle A K Q=A H D; φ = circular arc F v, and $\partial\varphi$=v r; z = elliptic arc, Q A, and ∂z = A B. Then per similar sectors, H v : v r :: H A : A B, that is, 1 : $\partial\varphi$ :: H A : ∂z, ther. $\partial z = \partial\varphi \times$ H A; but Prop. XXII. H A = a−2c+3c S² nearly, ther. $\partial z = \partial\varphi$ (a−2c +3c S²) = (a−2c) $\partial\varphi$+3c S² $\partial\varphi$; but S²=$\frac{1}{2}$(1−cos. 2φ), ther. ∂z=(a−2c) $\partial\varphi + \frac{3c}{2}\partial\varphi - \frac{3c}{2}$ cos. 2φ $\partial\varphi$. By integration z= $(a - \frac{c}{2})\varphi - \frac{3c}{4}$sin. 2$\varphi$ = a$\varphi - \frac{c}{2}(\varphi + \frac{1}{2}$ sin. 2φ), an expression which needs no correction.

Let M N be an arc of the elliptic meridian,

φ' the latitude of M, one of its extremities,

φ'' that of N, the other extremity; we have

$$E M = a\varphi' - \frac{c}{2}(\varphi' + \tfrac{1}{2} \sin. 2\varphi')$$

$$E N = a\varphi'' - \frac{c}{2}(\varphi'' + \tfrac{1}{2} \sin. 2\varphi'')$$

And their difference E N − E M, that is,

$$M N = a(\varphi'' - \varphi') - \frac{c}{2}\left\{(\varphi'' - \varphi') + \tfrac{1}{2}(\sin. 2\varphi'' - \sin. 2\varphi')\right\}.$$

In which equation a and c are the only unknown quantities. In like manner by the measurement of another arc of the meridian, another equation will be obtained, in which a and c are the only unknown quantities. Hence the values of a and c become known, from which the axes may be determined.

Thus. If l=length of an arc measured,

$m=\varphi''-\varphi'$=co-efficient of a,

$n=\frac{1}{2}\left\{(\varphi''-\varphi')+\frac{3}{2}(\sin.2\varphi''-\sin.2\varphi')\right\}$=co-efficient of c,

And if l'=length of another arc,

m'=co-efficient of a, and

n'=co-efficient of c; then we shall have

$m\,a-n\,c=l$, and $m'\,a-n'\,c=l'$; whence $a=\frac{n'\,l-n\,l'}{m\,n'-m'\,n}$, $c=\frac{m'\,l-m\,l'}{m\,n'-m'\,n}$, and $\frac{c}{a}=\frac{m'\,l-m\,l'}{n'\,l-n\,l'}$.

It is also evident that $c=\frac{m\,a-l}{n}=\frac{m'\,a-l'}{n'}$, and $b=a-c$.

Prop. XXIV.

A degree of the earth's equator is the first of two mean proportionals between the *last* and first degrees of latitude.

Let R and r represent the radii of curvature corresponding to a degree of latitude at the poles and at the equator, and C E the radius of the equator.

We have to prove that $C\,E=\sqrt[3]{r\,R^2}$.

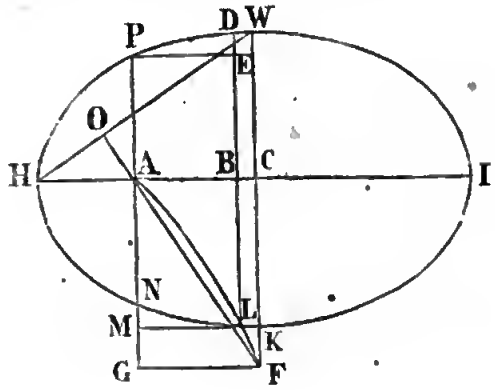
Per Cor. 1 and 2, Prop. VI. $R=\frac{C\,E^2}{C\,P}$, and $r=\frac{C\,P^2}{C\,E}$.

Then $\sqrt[3]{r\,R^2}=\sqrt[3]{\frac{C\,P^2}{C\,E}\times\frac{C\,E^4}{C\,P^2}}=\sqrt[3]{C\,E^3}=C\,E$, as required.

Prop. XXV.

If A be the center of curvature at H, the vertex of the transverse axis, and F, the center of curvature at W, the vertex of the

conjugate axis, and A F and H W be joined; the right-angled triangles H C W, A C F, will be similar.



By Cor. 1. and 2. Prop. VI. $HA = \frac{CW^2}{HC}$, and $WF = \frac{HC^2}{CW}$, ther. $HC - HA = AC = HC - \frac{CW^2}{HC} = \frac{HC^2 - CW^2}{HC}$; and $WF - WC = CF = \frac{HC^2}{CW} - CW = \frac{HC^2 - CW^2}{CW}$. Hence it will be, as $AC : CF :: \frac{HC^2 - CW^2}{HC} : \frac{HC^2 - CW^2}{CW} :: CW : CH$.

Therefore Def. 1. 6 : E. the triangles H C W, A C F are similar.

COROLLARY.

By producing F A to O, the triangles A O H, A C F will be similar, and F O at right angles to H W.

PROP. XXVI.

If the evolute A L F of the elliptic arc H P W intersect the ellipse in L, and the rectangle A C F G be completed, to determine the ordinate L M parallel to C H, when C F is greater than the semi-conjugate C K.

Produce G A to P; draw P E parallel, and L D perpendicular to H I. By Cor. 1. Prop. VI. $HA = \frac{CW^2}{HC}$ $\therefore$ $AI = \frac{2CH^2 - CW^2}{CH}$,

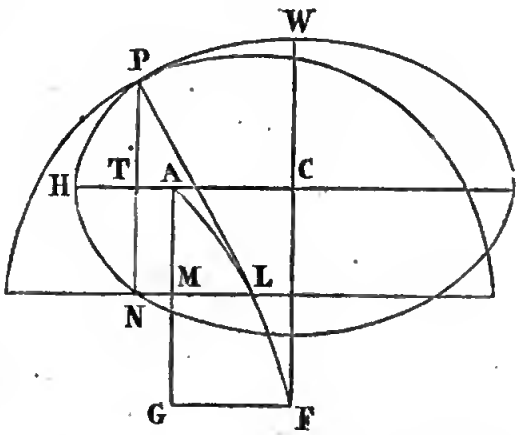
Then (16 Ellip.) $AP^2=BE^2=\frac{CW^4}{CH^4}(2CH^2-CW^2)=m^2$ ∴ $AP=m$. Put $AM=BL=BD=x$; then $x+m=LE$, and $x-m=ED$. Ther. (16 Ellip.) $PE^2=LM^2=\frac{HC^2}{CW^2}(x^2-m^2)=\frac{CH^4\cdot AM^2-CW^4(2CH^2-CW^2)}{CH^2\times CW^2}$.

Corollary.

Since the evolute A L F is always within the right-angled triangle A C F, it is evident that it will not intersect the ellipse when W F is less than W K.

Prop. XXVII.

To determine the co-ordinates A M, M L, at any point L of the evolute A L F.



Let L P represent the radius, and P N half the chord of curvature at any point P.

Put $HT=x$
$CH=a$
$CW=b$
$a^2-b^2=v^2$
Sin. P L N=s, and cos.=c.

Then per Prop. of ellipse $PT=\frac{b}{a}\sqrt{2ax-x^2}$.

By Cor. 1. Prop. VI. $HA = \frac{b^2}{a}$,

And Prop. XV. $LP = \frac{a^2 b^2}{(a^2 - v^2 s^2)^{\frac{3}{2}}} = \frac{a^2 b^2}{(b^2 + v^2 c^2)^{\frac{3}{2}}}$

Per trig. $PN = \frac{a^2 b^2 s}{(a^2 - v^2 s^2)^{\frac{3}{2}}}$, and $LN = \frac{a^2 b^2 c}{(b^2 + v^2 c^2)^{\frac{3}{2}}}$.

But $PN - PT = TN = AM$, and $LN + TH - HA = LM$.

That is, $\frac{a^2 b^2 s}{(a^2 - v^2 s^2)^{\frac{3}{2}}} - \frac{b}{a}\sqrt{2ax - x^2} = AM$,

And $\frac{a^2 b^2 c}{(b^2 + v^2 c^2)^{\frac{3}{2}}} - \frac{b^2}{a} + x = ML$.

Corollary 1.

When the angle PLN becomes 90 degrees, then $s=1$, $c=0$, $PT = CW = b$, $HT = HC = a$, $AM = CF$, and $LM = FG$. The two preceding expressions will then become $\frac{a^2}{b} - b = \frac{a^2 - b^2}{b} = CF$, and $a - \frac{b^2}{a} = \frac{a^2 - b^2}{a} = FG$. The same as determined in Prop. XXV.

Corollary 2.

If the angle PLN become nothing, then $s=0$, $c=1$, and $HT = x = O$; when this is the case, both AM and ML become nothing.

Corollary 3.

The length of the whole evolute $ALF = FW - AH = \frac{a^2}{b} - \frac{b^2}{a} = \frac{a^3 - b^3}{ab}$.

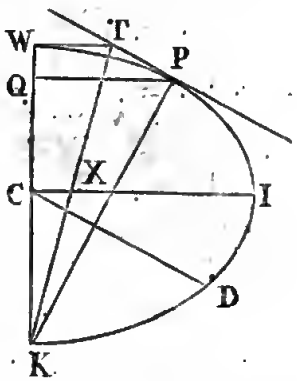
Corollary 4.

The length of any part of the evolute $AL = LP - AH = \frac{a^2 b^2}{(a^2 - v^2 s^2)^{\frac{3}{2}}} - \frac{b^2}{a} = \left\{\frac{a^2}{(a^2 - v^2 s^2)^{\frac{3}{2}}} - \frac{1}{a}\right\} b^2$.

The following Examples will show the usefulness of some of the foregoing Propositions.

Example 1.

To determine that point in the curve of a given ellipse, which is at the greatest distance from the extremity of the conjugate axis.—*Mathematical Repository*, No. 13.



Let K I W represent half the elliptic section, K W the conjugate, and C I the semi-transverse axis, draw the ordinate P Q; and K P at right angles to the tangent P T, intersecting C I in X; then will the angle P X I, or the angle K P Q, represent the latitude of the required point P, and K P* the required distance.

By Cor. 1. Prop. XXVIII. Ellip. Q K : Q C : : C I^2 : C K^2; by division Q K—Q C : Q K : : C I^2—C K^2 : C I^2; or C K : Q K : :

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* That K P must be at right angles to the tangent P T, may be thus demonstrated:

Let W T a tangent at W meet the tangent P T in T; draw the semi-diameter C D parallel to the tangent P T, and join K T. Then (Scholium to Prop. XIV. Ellip.) W T^2 : T P^2 : : C I^2 : C D^2, ther. (Prop. XXXV. Ellip.) W T is greater than T P, and as the angles at P and W are right angles, and K T common to both the triangles K P T and K W T, we have (47·1. E.) K W^2=K T^2—T W^2, and K P^2=K T^2—T P^2, but T W is greater than T P; therefore K P is greater than K W.

That is, K P (at right angles to the tangent P T) is greater than any other straight line that can be drawn from K to the curve.

$C I^2 - C K^2 : C I^2$, ther. $Q K = \frac{C K \times C I^2}{C I^2 - C K^2} = \frac{n m^2}{m^2 - n^2} = \frac{n m^2}{v}$;

by Prop. XXXVII. Ellip. $P Q = \frac{m^2}{\sqrt{m^2 + n^2 t^2}}$; (t=tang. K P Q):

Then per trig. $P Q : Q K :: \text{rad.} : \text{tang. } K P Q$. That is,

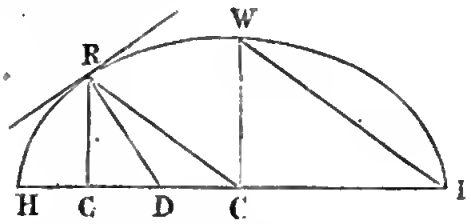
$\frac{m^2}{\sqrt{m^2 + n^2 t^2}} : \frac{n m^2}{v} :: 1 : t$; ther. $\frac{t^2}{m^2 + n^2 t^2} = \frac{n^2}{v^2}$, or $t =$

$\frac{m n}{\sqrt{v^2 - n^4}} = \frac{m n}{\sqrt{(m^4 - 2 m^2 n^2 + n^4 - n^4)}} = \frac{n}{\sqrt{m^2 - 2 n^2}} =$

$\frac{C K}{\sqrt{C I^2 - 2 C K^2}}$. Hence $K P = \frac{C I^2}{\sqrt{(C I^2 - C K^2)}}$.

Example 2.

If the earth be an oblate spheroid, the ratio of whose axes is 230 to 229, in what latitude will the line of motion, in which a heavy body falls to the earth, make the greatest angle with one drawn perpendicular to the surface, and how much is that angle?—*Lady's Diary*, 1806.



Let H W I represent half the elliptic meridian passing through the required latitude R; H I the equatorial, and C W half the polar axis, and join I W. Then by Proposition XL. Ellip. we have the following construction and computation.

Construction.

Draw C R parallel to I W, and draw the normal R D, and D R C will be the required angle.

Computation.

$$\text{Tang. } D R C = \frac{C I^2 - C W^2}{2 C I \times C W} = \frac{C I + I W}{2 C I \times C W} = (C I - C W = I)$$

$\frac{230+229}{2\times230\times229} = \frac{459}{105340} = \frac{1}{229{\cdot}5}$, corresponding to 14′. 58″$\frac{1}{4}$. But the angles G R C and G R D are complements to each other; therefore $90°-C\,R\,G = G\,R\,D = 90°-(C\,R\,D+D\,R\,G)$, that is, $2\,G\,R\,D = 90°-C\,R\,D$; or $G\,R\,D = 45°-\frac{1}{2}C\,R\,D$, therefore $90°-G\,R\,D = G\,D\,R = 90°-45°+\frac{1}{2}C\,R\,D = 45°+\frac{1}{2}C\,R\,D$ $= 45°.7'.29''.22'''\frac{1}{2}$, the required latitude.

Example 3.

To determine a place on the earth, where a degree of the meridian is equal to a degree of the equator; the ratio of the axis to the equatorial diameter being that of 229 to 230.—*Lady's Diary*, 1802.

Put $230=a$, $229=b$, $a^2-b^2=v^2$, $S=$ nat. sin. of the required latitude, and R the radius of curvature of the meridian in the proposed latitude.

Then by Prop. XV. $R = \frac{a^2\,b^2}{(a^2-v^2\,S^2)^{\frac{3}{2}}}$; but the lengths from the degrees being as their respective radii of curvature, and from the conditions of the question, we have $\frac{a^2\,b^2}{(a^2-v^2\,S^2)^{\frac{3}{2}}} = a$; from whence $S^2 = \frac{a^2-(a\,b^2)^{\frac{2}{3}}}{a^2-b^2} = \frac{a^2-(a\,b^2)^{\frac{2}{3}}}{a+b} = (a-b=1)\frac{306}{459}$ ∴ Log. $S =$ Log. $\sqrt{\frac{306}{459}} = \bar{1}{\cdot}9119544$, corresponding to 54°. 44′. 8″, the required latitude.

Example 4.

Given the earth's axes, 7977 and 7940 miles, to determine, by a simple and accurate process, the radius of curvature in apparent latitude 50°. 23′.—*Gentleman's Diary*, 1805.

Put $7977=a$; $7940=b$, $a^2-b^2=v^2$, and nat. sin. 50°. 23′$=S$; then (Prop. XV.) $2\,R = \frac{a^2\,b^2}{(a^2-v^2\,S^2)^{\frac{3}{2}}} = \frac{63632529\times63043600}{(63283059)^{\frac{3}{2}}} =$

7968·9. Ther. $R = \frac{7968{\cdot}9}{2} = 3984{\cdot}45$ = the required radius of curvature.

In Prop. XV. a and b represent the semi-axes; but in this example they represent the whole axes; ther. 2 R is used instead of R.

Example 5.

To find in what latitude a degree of the meridian is equal in length to two degrees of longitude in the same latitude, the earth being considered as an oblate spheroid, the axes of which are as 229 to 230.—*Lady's Diary*, 1810.

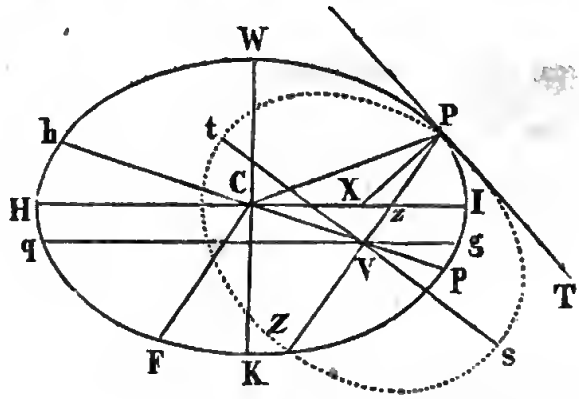
Put 230=a, 229=b, $a^2-b^2=v^2$; S, C, T=sin. cos., and tang. of the required latitude; R = radius of curvature of the meridian, and r=radius corresponding to a degree of longitude.

By Prop. XXXVII. $r = \frac{a^2}{\sqrt{a^2+b^2\,t^2}}$, by Prop. XV. $R = \frac{a^2\,b^2}{(b^2+v^2\,C^2)^{\frac{3}{2}}}$. But from the lengths of the degrees being as their respective radii of curvature, and from the condition of the question, we have $\frac{a^2\,b^2}{(b^2+v^2\,C^2)^{\frac{3}{2}}} = \frac{2\,a^2}{(a^2+b^2\,T^2)^{\frac{1}{2}}}$. But $a^2+b^2\,T^2 = a^2+\frac{b^2\,S^2}{C^2} = \frac{a^2\,C^2+b^2\,(1-C^2)}{C^2} = \frac{b^2+(a^2-b^2)\,C^2}{C^2} = \frac{b^2+v^2C^2}{C^2}$; ther. $\frac{b^2}{(b^2+v^2\,C^2)^{\frac{3}{2}}} = \frac{2}{\frac{1}{C}\,(b^2+v^2\,C^2)^{\frac{1}{2}}}$, or $\frac{b^2}{(m^2)^{\frac{3}{2}}} = \frac{2\,C}{(m^2)^{\frac{1}{2}}}$, that is $\frac{b^2}{m^3} = \frac{2\,C}{m}$. $\therefore\ 2\,C\,m^3=b^2$; or $2\,C\times(b^2+v^2\,C^2)=b^2$; ther. $v^2\,C^3+b^2\,C=\frac{b^2}{2}$. In a sphere $V^2=o$; $\therefore\ C=\frac{1}{2}=\cos.\,60°$. But if, instead of the cos. C, we substitute its equal $\frac{1}{\sec.}=\frac{1}{\varphi}$, the above equation will become $\frac{v^2}{\varphi^3}+\frac{b^2}{\varphi}=\frac{b^2}{2}$. $\therefore\ \varphi^3-2\,\varphi^2=\frac{2\,v^2}{b^2}=\frac{2(a^2-b^2)}{b^2}=\frac{a+b}{b^2}\times 2(a-b=1)$; ther. $(\varphi-2)\times\varphi^2=\frac{918}{52441}$

$= 0{\cdot}0175053$; hence $\varphi = 2{\cdot}0043559 =$ sec. 60°.4′.18″.7. In a sphere $\frac{230}{229} = 1$; ther. $\varphi^3 - 2\varphi^2 = 0$; hence $\varphi = 2 =$ sec. 60°.

EXAMPLE 6.

If an ellipsoid, whose revolving semi-axe is 8, and its fixed semi-axe $5\frac{1}{2}$ inches, be cut by a plane in latitude 50°.22′.18″ at right angles to the meridian, and inclined to the horizon in an angle of 70°; to determine the area of the section made by the cutting plane, and the solidity of the separated parts.



Let H W I K represent the plane of the meridian through P, the given latitude, t Z s P the elliptic section, P Z the common section of the two planes, forming with the tangent P T an angle Z P T equal to 70°, intersecting H I in z; draw P X at right angles to P T meeting H I in X, draw the semi-diameter C F parallel to P Z; through V the center of P Z draw q V g parallel to the transverse axis H I, whose conjugate is W K; join C V, which produce to intersect the meridian in h and p, and join C P.

By Prop. XXXVII. Ellip. $PX = \frac{CK^2}{C\sqrt{CH^2 + T^2 \cdot CK^2}} =$ 4·5573; C = cos. and T = tang. P X I = 50°.22′.18″; then in the triangle P X z, are given P X and all the angles, to find X z = 1·651, and P z = 3·718; $CX = \frac{CH^2 - CK^2}{\sqrt{CH^2 - T^2 \cdot CK^2}} = 3{\cdot}243$ (ibid.), therefore z I = 3·106, and H z = 12·894, then per Scho-

lium to Prop. XIV. Ellip. we have $Hz \times zI : Pz \times zZ :: CI^2 : CF^2 = \frac{CH^2 \times CK}{c^2(CK^2+t^2.CH^2)}$ (ibid.) c=cos. t=tang. HCF=PzI= 70°.22′.18″ = (5·6716)² ∴ $zZ = \frac{Hz \times zI \times CF^2}{Pz \times CI^2} = 5{\cdot}414$. Therefore Pz+zZ=PZ=9·132; and PV=VZ=4·566.

Again per Scholium to Prop. XIV. Ellip. we have $qV \times Vg : PV \times VZ :: CI^2 : CF^2$ ∴ $qV \times Vg = \frac{PV \times VZ \times CI^2}{CF^2} = \frac{PV^2 \times CI^2}{CF^2}$, but from the nature of the circle, qV×Vg=square of the ordinate corresponding to the point V, or in this example to the square of the semi-transverse axis of the elliptic section tZsP; ther. $Vt = Vs = \sqrt{\frac{PV^2 \times CI^2}{CF^2}} = 6{\cdot}4405$, and thence the transverse st=12·881, whose conjugate axis is PZ. In the triangle CVz we have PV−Pz=Vz=0·748, Cz=4·894 and the angle CzV=70°.22′.18″, to find CV=4·7318; and Prop. XLV. Ellip. $Cp = \frac{CV \times CI}{\sqrt{CI^2-(Vs)^2}} = 7{\cdot}9769$, hence Vh=12·709 and Vp=3·245. And from Dr. Hutton's Mensuration,

The area of the section PtZs	=	92·368
Solidity of greater segment ZHWP	=	1316·3
Ditto - - less segment ZpIP - -	=	158·2
Whole spheroid - - - - - - - - -	=	1474·5

} inches.

Example 7.

In latitude 50°.41′ north the length of a degree on the meridian is 60851 fathoms, and the measure of a degree perpendicular to the meridian is 61182 fathoms, to find the earth's axes.—*Trig. Survey*, vol. i. p. 309.

Put s=nat. sin. 50°.41′, m=60851, p=61182, and d=57·29, &c. the degrees in a circular arc equal to radius. Let H = an arc corresponding to the nat. sin. of $s\sqrt{\frac{m}{p}}$; then (cor. 3. Prop. XVI.)

$$d\sqrt{p\,m} \times \frac{\cos.^2 \text{lat.}}{\cos.^2 H} = \tfrac{1}{2} \text{ the polar axis,}$$

$$d\,p \times \frac{\cos. \text{lat.}}{\cos. H} \ldots = \tfrac{1}{2} \text{ the equatorial axis;}$$

Log. d = 57·29, &c.	= 1·7581226
Log. $\sqrt{p\,m} = \sqrt{(61182 \times 60851)}$. . .	= 4·7854457
2 log. cos. 50°. 41′.	= $\bar{1}$·6036384
	6·1472067
Log. $\sqrt{\frac{m}{p}} = \sqrt{\frac{60851}{61182}}$ . . .	= $\bar{1}$·9988220
Log. sin. 50°. 41′	= $\bar{1}$·8885479
Log. sin. of H	= $\bar{1}$·8873699
2 log. cos. of H, subtract	= $\bar{1}$·6071274
Diff.=log. 3468002 fathoms=½ polar axis	= 6·5400793

Log. d	= 1·7581226
Log. p = 61182.	= 4·7866237
Cos. lat.	= $\bar{1}$·8018192
	6·3465655
Log. cos. of H, subtract	= $\bar{1}$·8035637
Diff.=3491417 fathoms=½ equatorial axis	= 6·5430018

EXAMPLE 8.

Having at the same point the length of a degree on the meridian 60850 fathoms, and also that of a degree perpendicular to the meridian 61182 fathoms, to find the length of a degree forming an angle with the meridian equal to 81°. 56′. 53″.—*Lady's Diary*, 1801.

Put p = 61182, m = 60850, and s = nat. sin. 81°. 56′. 53″. Then (Cor. 2. Prop. XVII.)

Log. (p − m) = 332 . . . . . = 2·5211381
Log. p = 61182 . . . . . = 4·7866237

Log. $\frac{p-m}{p}$ . . Subtract . . . . = $\bar{3}$·7345144

Log. $\sqrt{\frac{p-m}{p}}$ . . . . . = $\bar{2}$·8672572
Log. sin. 81°.56′.53″ . . Add . . = $\bar{1}$·9956973

Log. sin. of an arc A . . . . . = $\bar{2}$·8629545

2 log. cos. of A . . . . . = $\bar{1}$·9976834
Log. m = 60850 . . . . . = 4·7842606

Diff. = log. 61175·45 . . . . . = 4·7865772

Therefore the length of the required oblique degree is 61175·45 fathoms.

Example 9.

In latitude 51°.28′.40″ N. the measure of a degree on the meridian is 60868 fathoms, and in latitude 51°.5′ N. the length of a meridional degree is 60859 fathoms; required the earth's axes.—*Mathematical Repository*, No. 14.

Then (A) Prop. XVIII.

Log. $\sqrt[3]{\frac{d}{D}}$ = log. $\sqrt[3]{\frac{60859}{60868}}$ . . . = $\bar{1}$·9999786
Log. sin. B = 51°.5′ . . . . . = $\bar{1}$·8910133

Log. sin. E = 51°.4′.47″.24‴ . = $\bar{1}$·8909919

Log. $\sqrt[3]{\frac{d}{D}}$ . . . . . = $\bar{1}$·9999786
Log. cos 51°.5′ . . . . . = $\bar{1}$·7980906

Log. cos. F = 51°.5′.8″.11‴ . . = $\bar{1}$·7980692

Log. sin. A+B = 77°. 26′. 20″. 0‴ = 9·9894786
Log. sin. A−B = 0 . 23 . 40 . 0 = 7·8378598
Ar. comp. A+E = 102 . 33 . 27 . 24 = 0·0105155
Ar. comp. A−E = 0 . 23 . 52 . 36 = 2·1583037

2) 19·9961576

9·9980788

A+B . . . . . = 9·9894786
A−B . . . . . = 7·8378598
Ar. comp. A+F = 102 . 33 . 48 . 11 = 0·0105253
Ar. comp. A−F = 0 . 23 . 31 . 49 = 2·1646502
57 . 29, &c. . . = 1·7581226
d = 60859 . . . . = 4·7843248

OE = 3491725 . . . = 6·5430401

Log. sin. A+B. . . . . = 9·9894786
Log. sin. A−B. . . . . = 7·8378598
Ar. comp. A+F . . . . . = 0·0105253
Ar. comp. A−F. . . . . = 2·1646502

2) 0·0025139

0·0012569

A+B . . . . . = 9·9894786
A−B . . . . . = 7·8378598
Ar. comp. A+E . . . . . = 0·0105155
Ar. comp. A−E . . . . . = 2·1583037
57·29, &c. . . . . = 1·7581226
d . . . . . = 4·7843248

OK = 3466266 . . . . = 6·5398619

Log. sin. A+F . . . . . = 9·9894747
Log. sin. A−F . . . . . = 7·8353498
Ar. comp. A+E . . . . . = 0·0105155
Ar. comp. A−E . . . . . = 2·1583037

2) 19·9936437

$\frac{OK}{OE} = 0{\cdot}9927086 \ldots\ldots = 9{\cdot}9968218$

OK : OE :: 0·9927086 : 1 :: 231 : 232·696
OK : OE :: 1 : 1·0073449 :: 231 : 232·696

Or OK ÷ OE may be thus:

From log. OK . . . . . = 6·5398619
Take log. OE . . . . . = 6·5430401

Remain same logarithm as above . . . $\bar{1}{\cdot}9968218$

EXAMPLE 10.

In latitude 66° . 20′ N. the measure of a degree on the meridian is 57438 toises, and the measure of a degree at the equator is 56757 toises, to find the earth's axes.—*Robertson's Navigation.*

ARTICLE (C) PROP. XVIII.

Log. d = 56757 . . . . = 4·7530194
Log. D = 57438 . . . . = 4·7591993

3) $\bar{1}{\cdot}9938201$

Cos. F = 5°. 34′. 34″ . . . . = $\bar{1}{\cdot}9979400$
A = 66 . 20 . 0

71 . 54 . 34 = A+F
60 . 45 . 26 = A−F

Log. sin. A+F . . . . . = 9·9779827
Log. sin. A−F . . . . . = 9·9407941

2) 19·9187768

Log. $(S.(A+F)\ S.(A-F))^{\frac{1}{2}}$ = 9·9593884 (a)
Sin. A, subtract . . . . . = 9·9618463

Log. $\frac{CW}{CH} = 0{\cdot}9943565$ . . = $\bar{1}{\cdot}9975421$

Log. sin. A	= 9·9618463
Log. 57·29; &c.	= 1·7581226
Log. d	= 4·7530194
	16·4729883
Log. (a)	= 9·9593884
Log. C W in toises	= 6·5135999
Add*	0·0276553
Log. C W = 3477405 fath. .	= 6·5412552
2 log. A	= 19·9236926
Log. 57·29, &c.	= 1·7581226
Log. d	= 4·7530194
	26 4348346
2 log. (a)	= 19·9187768
Log. C H in toises	= 6·5160578
Add	0·0276553
Log. C H = 3497140 fath. .	= 6·5437131

---

* Fath. : Toise :: 6 : 6·3945. Doctor Hutton's Dictionary.

Ther. Fath. × 6 = Toise × 6·3945.

Or Fath. $= \frac{6{\cdot}3945}{6} \times$ Toise,

Toise $= \frac{6}{6{\cdot}3945} \times$ Fath.

Log. $\frac{6{\cdot}3945}{6}$ . . . . . = 0·0276553

Log. $\frac{6}{6{\cdot}3945}$ . . . . . = $\bar{1}$·9725447

SAME EXAMPLE, BY PROP. XIX.

Log. 57438	= 4·7591993
Log. 56757	= 4·7530194
	3) 0·0061799
	0·0020599
Log. cos. 66° . 20′	= 9·6035936
Cos. A=66° . 12′ . 50″ . . .	= 9·6056535
Log. tang. A	=10·3557948
Log. tang. 66° . 20′	=10·3582527
Log. $\frac{C W}{C H}$	= $\bar{1}$·9975421
Tang. lat.	=10·3582527
Log. 57·29, &c.	= 1·7581226
Log. 56757	= 4·7530194
	16·8693947
Log. tang. A	=10·3557948
Log. C W, in toises . . .	= 6·5135999
2 tang. lat.	=20·7165054
Log. 57·29, &c.	= 1·7581226
Log. 56757	= 4·7530194
	27·2276474
2 log. tang. A	=20·7115896
Log. C H, in toises	= 6·5160578

Each of the preceding methods brings out exactly the same answers, although the principles from whence they are derived are different.

Example 11.

If a = half the earth's equatorial diameter, and b = half the polar axis, and $\frac{a-b}{a} = \frac{1}{300}$; to find the ratio of b to a.

From the given equation we have $(300-1) \times a = 300 \times b$. Therefore $b : a :: 299 : 300 :: 1 : \frac{300}{299} :: 1 : 1 + \frac{1}{299} :: 1 : 1 + 0{\cdot}0033445$.

Example 12.

Given the earth's compression $\frac{1}{300}$, and the measure of a degree of latitude at the equator 56747 toises, to find the length of a meridional degree in latitude 30°.—*Vince's Astronomy*, vol. ii. page 110.

By Cor. 3. Prop. XX. $D = \frac{d}{1 - 3\,e\,S^2}$, where $d = 56747$, $3e = \frac{1}{299}$ (exam. 11.), and $S^2 = \frac{1}{4}$. Therefore $D = \frac{56747}{1 - \frac{3}{1196}} = \frac{1196 \times 56747}{1193} = \frac{67869412}{1193} = 56889$ toises, the required length of the meridional degree.

Example 13.

From the data given in Example 9th, to find the ratio of the earth's diameters by Prop. XXI.

S = Sin. 51° . 28′ . 40″ - - log. = $\bar{1}{\cdot}8934103$
s = Sin. 51 . 5 . 0 - - log. = $\bar{1}{\cdot}8910133$
D = 60868 fathoms - - - log. = $4{\cdot}7843890$
d = 60859 fathoms - - - log. = $4{\cdot}7843248$
D − d = 9 do. ar. com. - log. = $\bar{1}{\cdot}0457575$. A. C.
- - - - - - - - - - - - - - - log. 3 = $0{\cdot}4771213$

A = an arc corresponding to the sin. $\frac{s}{S}\sqrt{\frac{d}{D}}$,

And $\frac{3\,S^2\,D - 3\,s^2\,d}{D-d} = \frac{\text{Cos.}^2\,A \times S^2 \times 3\,D}{D-d}$.

First to find the arc A.

Log. d . . . . . = 4·7843248
Log. D. ar. comp. . . . . = $\bar{5}$·2156110

2) $\bar{1}$·9999358

Log. $\sqrt{\frac{d}{D}}$ . . . . . = $\bar{1}$·9999679
Log. s . . . . . = $\bar{1}$·8910133
Log. S. ar. comp. . . . . = 0·1065897

Log. sin. A=83°. 56′. 44″. . = $\bar{1}$·9975709

To find $\frac{3S^2D-3s^2d}{D-d}$.

Log. $\cos.^2$ A . . . . . = $\bar{2}$·0462862
Log. S^2 . . . . . = $\bar{1}$·7868206
Log. 3 . . . . . = 0·4771213
Log. D . . . . . = 4·7843890
Log. D−d, ar. comp. . . . = $\bar{1}$·0457575

Log. 138·16 . . . . . = 2·1403746

Hence C K : C H :: 138·16 : 139·16 :: 231 : 232·672.
And C H : C K :: 138·16 :: 137·16 :: 232·696 : 231·011.
Which very nearly agrees with Example 9th.

EXAMPLE 14.

Given the earth's compression $\frac{1}{300}$, and the measure of a degree of latitude at the equator 56747 toises, to find the length of a meridional degree in latitude 73°, by Cor. to Prop. 21.

$D=\frac{d}{\cos.^2 B}=\frac{56747}{\cos.^2 B}$; $B=\sqrt{\frac{1}{299}}\times$ sin. 73°.

Log. 3 . . . . . 0·4771213
Log. 299. ar. comp. . . . . $\bar{3}$·5243288

2) $\bar{2}$·0014501

$\bar{1}$·0007250
Log. sin. 73° . . . . . = $\bar{1}$·9805963

Log. sin. 5°. 29′. 48″ . . . = $\bar{2}$·9813213

Cos. 5° . 29′ . 48″ . . . . . = $\bar{1}$·9979984
2

$\bar{1}$·9959968

Ar. comp. . . . . = 0·0040032
Log. 56747 . . . . . = 4·7539429

Log. 57272·5 . . . . . = 4·7579461

Therefore the length of a meridional degree in lat. 73° . . . . . = 57272·5 toises
By actual multiplication and division . . . = 57272·5194
By Vince's Astronomy, vol. ii. page 110 . . = 57269

EXAMPLE 15.

From the data given in Example 4th to find the radius of curvature, by Cor. to Prop. 22nd.

$R = \frac{a-2c}{\cos.^2 A}$; a=7977, b=7940, a−b=c=37,

A=an arc, corresponding to $S\sqrt{\frac{3c}{a}}$.

Log. 3c = 111 . . . . . = 2·0453230
Log. a = 7977. . . . . = 3·9018396

2) $\bar{2}$·1434834

$\bar{1}$.0717417
Log. sin. 50° . 23′ . . . . = $\bar{1}$·8866756

Log. sin. 5° . 12′ . 49″ . . . = $\bar{2}$·9584173

Cos. 5° . 12′ . 49″. . . . . = $\bar{1}$·9961995
2

$\bar{1}$·9963990

Ar. comp. . . . . = 0·0036010
Log. a−2c=7903 . . . . = 3·8977920

Log. 7968·8 . . . . . = 3·9013930

Ther. $R = \frac{7968{\cdot}8}{2} = 3984{\cdot}4$ = the required radius of curvature. By Example 4th, the radius is 3984·45.

In this Example, a and b represent the whole axes, therefore 2 R is used instead of R.

EXAMPLE 16.

Given the lengths of two meridional degrees at the equator, and in lat. 45°. N. 68·732 and 69·092 English miles respectively, to find the compression.

Let $\sqrt[3]{\frac{68{\cdot}732}{69{\cdot}092}} = v$. Then (Cor. 3. Prop. XVIII.) $\frac{OE-OK}{OE} = \frac{(1+v)(1-v)}{2\sin.^2 45°}$.

Log. 68·732	= 1·8371590
Log. 69·092 (a · C)	= $\bar{2}$·1605722
	3) $\bar{1}$·9977312
Log. v = ·99826	= $\bar{1}$·9992437
Log. 1+v = 1·99826 . . .	= 0·3006520
Log. 1−v = 0·00174 . . .	= $\bar{3}$·2405492
Log. 2 (a · c)	= $\bar{1}$·6989700
Log. sin.² 45° (a · c)	= 0·3010300
	$\bar{3}$·5412012
Log. 287·6 arith. comp. . . .	2·4587988

Therefore the compression is $\frac{1}{287{\cdot}6}$. By Cor. 2. Prop. XVIII. The compression is $\frac{1}{287}$.

OF

CENTRIPETAL AND CENTRIFUGAL FORCES

IN

ELLIPTICAL ORBITS.

CENTRIPETAL FORCES.

DEFINITION I.

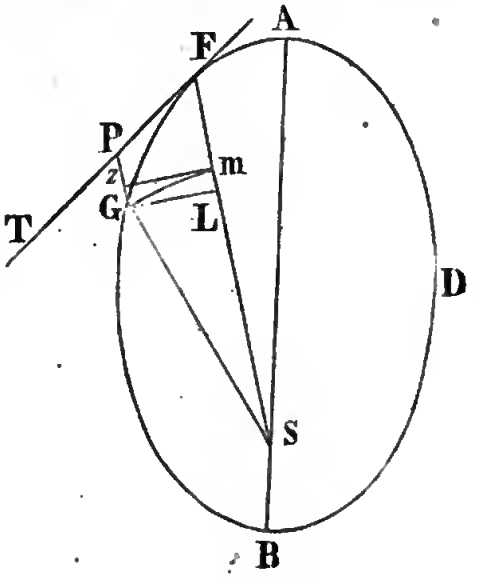
A CENTRIPETAL FORCE is that by which a moving body is continually drawn from its rectilinear motion, and is perpetually solicited towards some center.

DEFINITION II.

A *Centrifugal Force* is the resistance made by a moving body to its being turned out of its rectilinear motion. And because action and re-action are equal and contrary, it follows that in a circle the centripetal and centrifugal forces are equal.

DEFINITION III.

The *centrifugal* force of a body at any point in its orbit, is equal to the *centripetal* force with which the body would describe a *circle* at the same distance, and with the same velocity, in a direction perpendicular to that distance.



Thus, let A F B D be an elliptic orbit described by a body about the focus S; let F G be described in an indefinitely little part of time; join S F, S G, and draw the tangent F T, draw G P parallel, and G L perpendicular to S F; with the center S, and distance S G describe the circular arc G m, and draw m z parallel to G L. Then will G P represent the centripetal, and L m=G z, the centrifugal forces at the point F.

DEFINITION IV.

If a body describe an orbit round an immovable center of force, the areas described by lines drawn from the body to the center, are in the same plane, and proportional to the times.

PROPOSITION I. FIG. 1.

If the ellipses A G D, a g d, be described about the common focus S, and S Q, S q be drawn perpendicular to the tangents F P, f p; vel. at F : vel. at f :: $\frac{\text{area S F G}}{\text{S Q}} : \frac{\text{area S f g}}{\text{S q}}$.

Let F G, f g, be described in indefinitely little equal parts of time; and join S F, S G, S f, S g; then will S Q × F G : S q × f g :: area S F G : area S f g; ther. $\frac{\text{area S F G}}{\text{S Q}} : \frac{\text{area S f g}}{\text{S q}}$ :: F G : f g :: vel. at F : vel. at f.

COR. 1. In the same orbit, S F G=S f g; in that case, vel. at F : vel. at F :: $\frac{1}{\text{S Q}} : \frac{1}{\text{S q}}$.

COR. 2. When F, f, coincide with A, a, S Q=S A, and S q=S a; then, vel. at A : vel. at a :: $\frac{\text{area S A G}}{\text{S A}} : \frac{\text{area S a g}}{\text{S a}}$ :: $\frac{1}{\text{S A}} : \frac{1}{\text{S a}}$ (in the same orbit).

COR. 3. If F, f, coincide with B, we then have vel. at B :

vel. at b :: $\frac{\text{area SBG}}{SB} : \frac{\text{area Sbg}}{Sb} :: \frac{1}{SB} : \frac{1}{Sb}$ (in the same orbit).

Cor. 4. If F, f, coincide with N, n, then $SQ = CN$, and $Sq = cn$; we shall then have vel. at N : vel. at n :: $\frac{\text{area SNG}}{CN} : \frac{\text{area Sng}}{cn} :: \frac{1}{CN} : \frac{1}{cn}$ (in the same orbit). AB, ND, ab, nd, the transverse and conjugate axes of the ellipses AGD, agd; and G, g, are always supposed to be indefinitely near to F and f.

Prop. II. Fig. 1.

If a body revolve in an ellipse AGD, to find the centripetal force tending to the focus S.

Let S be the focus of the ellipse, AB the greater axes, C the center, and CD half the less axis; let FG, fg, be described in indefinitely equal parts of time; join SF, SG, Sf, Sg; draw SQ perpendicular to the tangent at F, draw GP parallel to SF, and CI parallel to FP; let FEH be the circle of curvature at F, and FE the chord passing through the focus.

The centripetal force $PG \propto \frac{FG^2}{FE} \propto \frac{\text{vel.}^2}{FE}$ (1. cir. cur.); but (Cor. 1. Prop. I.) $\text{vel.}^2 \propto \frac{1}{SQ^2}$ $\therefore PG \propto \frac{1}{FE \times SQ^2}$; but (31 Ellip.) $SQ^2 = \frac{SF^2 \times CD^2}{CI^2}$, and (3. cir. cur.) $FE = \frac{2CI^2}{AC}$; ther. $PG \propto \frac{CA}{2CI^2} \times \frac{CI^2}{SF^2 \times CD^2} \propto \frac{1}{SF^2} \times \frac{CA}{2CD^2} \propto \frac{1}{SF^2}$, because CA and CD are given.

Therefore, centripetal force at F : centrip. force at f :: $\frac{1}{SF^2} : \frac{1}{Sf^2}$.

Prop. III. Fig. 2.

If a body revolve in an ellipse AGD, to find the centripetal force tending to the center C.

Let C be the center of the ellipse, A B the greater axis, and C D half the less axis; let F G, f g, be described in indefinitely little equal parts of time; join C F, C G, C f, C g; draw C Q perpendicular to the tangent at F; draw G P parallel to C F, and C I parallel to F P; let F E H be the circle of curvature at F, and F E the chord of curvature passing through the center C.

By Prop. II. $PG \propto \frac{1}{FE \times CQ^2}$, and (Cor. 1. 31. Ellip.) $CQ^2 = \frac{AC^2 \times CD^2}{CI^2}$; and (1. cir. cur.) $FE = \frac{2CI^2}{CF}$; therefore $PG \propto \frac{CF}{2CI^2} \times \frac{CI^2}{CA^2 \times CD^2} \propto CF \times \frac{1}{2CA^2 \times CD^2} \propto CF$, because C A and C D are given; therefore centrip. force at F : centrip. force at f :: C F : C f.

Prop. IV. Fig. 1.

If a body be projected from a given point F, in a given direction F P, with a given velocity, and at the same time be acted upon by a centripetal force, which varies inversely as the squares of the distances from S, it will describe an ellipse whose focus is S.

Let S be the center of force, F the given point in the curve, and F P a tangent at F, the given direction; Let F G be described in an indefinitely little part of time; draw S F, S G; draw G P parallel to S F, and S Q perpendicular to F P; and let L represent the latus-rectum.

If the curve be an ellipse, we have (Prop. IV. V. cir. cur.) $SF^2 : SQ^2 :: \frac{GF^2}{GP} : L$; but (Prop. II.) $GP \propto \frac{1}{SF^2}$; therefore $SF^2 : SQ^2 :: GF^2 \times SF^2 : L = SQ^2 \times GF^2$; but S P and the angle S F Q are given, therefore S Q may be found; and F G (being indefinitely little) will represent the velocity, which is also given; therefore L, the latus-rectum, is given. Hence we have sufficient data to describe the ellipse; therefore the body will revolve in the ellipse with those data; for it cannot describe any other curve with the same data. For a construction see Prop. XLVI. Ellipse.

Prop. V. Fig. 1.

Bodies describing ellipses A G D, a g d, about the same common focus S, with forces which vary inversely as the squares of the distances from the focus, will have their principal latera-recta to each other, in the duplicate ratio of the areas described in the same time.

Let F G, f g, be described in indefinitely little equal parts of time, join S F, S G, S f, S g; draw G L, g l, perpendicular, and G P, g p, parallel to S F, S f, respectively; put L=latus-rectum of A G D, and l=latus-rectum of a g d.

Then (area S F G)^2 : (area S f g)^2 :: S F^2 × G L^2 : S f^2 × g l^2;

But (5. cir. cur.) G L^2=G P × L, and g l^2=g p × l;

But (Prop. II.) G P $\propto \frac{1}{S F^2}$, and g p $\propto \frac{1}{S f^2}$;

Therefore (area S F G)^2 : (area S f g)^2 :: L : l.

Cor. If one of the orbits (a g d) be a circle, then l=diameter of that circle. In that case,

Area in the ellipse : area in the circle :: $\sqrt{\text{Latus-rectum}}$: $\sqrt{\text{Diam.}}$

Prop. VI. Fig. 1.

The *squares* of the periodic times in all ellipses described about the same common focus, are as the *cubes* of their transverse axes.

Let A B, N D, a b, n d, be the transverse and conjugate axes of the two ellipses A G D, a g d, described about the common focus S; let F G, f g, be described in indefinitely little equal parts of time; join S F, S G, S f, S g; let P represent the periodic time in the orbit A N B D, and p the periodic time in the orbit a n b d; put L and l, for their respective latera-recta.

Conceive the areas of these ellipses to be made up of the indefinitely little triangles S F G, S f g; and as these areas are proportional to the times of describing, it will be

P × S F G : p × S f g :: A N B D : a n b d;

But (Prop. V.), (S F G)^2 : (S f g)^2 :: L : l,

Ther. $P^2 \times L : p^2 \times l :: AB^2 \times DN^2 : ab^2 :: dn^2$,

That is, $P^2 : p^2 :: \frac{AB^2 \times DN^2}{L} : \frac{ab^2 \times dn^2}{l} :: AB^3 : ab^3$.

Cor. 1. The periodic times are as the rectangles of their axes directly, and subduplicate ratio of their latera-recta inversely.

For, $P. p :: \frac{AB \times DN}{\sqrt{L}} : \frac{ab \times dn}{\sqrt{l}}$.

Cor. 2. If the ellipses become circles, the transverse and conjugate diameters, and the latera-recta become equal to each other; then the squares of the periodic times will be as the cubes of the diameters.

Cor. 3. If one of the ellipses (a b n d) become a circle, the diameters of which are equal to the transverse diameter AB; the periodic times in the ellipse and circle will be equal.

Prop. VII. Fig. 1.

If in bodies describing ellipses AGD, agd, about the same common focus S, with forces which vary inversely as the squares of the distances from the focus, SQ, Sq, are perpendiculars to the tangents at F, f, and L, l, are put for the principal latera-recta, then

Vel. at F : vel. at f :: $\frac{\sqrt{L}}{SQ} : \frac{\sqrt{l}}{Sq}$.

Let FG, fg, be described in indefinitely little equal parts of time; join SF, SG, Sf, Sg; then FG, fg, will represent the velocities at the points F, f.

And $SFG : Sfg :: FG \times SQ : fg \times Sq :: \sqrt{L} : \sqrt{l}$ (Prop. V.);

Ther. $\frac{\sqrt{L}}{SQ} : \frac{\sqrt{l}}{Sq} :: FG : fg ::$ vel. at F : vel. at f.

Cor. 1. The latera-recta are to each other as vel.^{2} × perpend.^{2} For, $FG^2 \times FQ^2 : fg^2 \times fq^2 :: L : l ::$ (vel.^{2} at F) × SQ^2 : (vel.^{2} at f) × Sq^2.

Cor. 2. If the ellipses become circles, then L=2AC, SQ=

AC, l=2ac, Sq=ac; and the analogy in the Proposition becomes,

Vel. at F : vel. at f :: $\frac{\sqrt{2AC}}{AC} : \frac{\sqrt{2ac}}{ac} :: \frac{1}{\sqrt{AC}} : \frac{1}{\sqrt{ac}}$;

AB, ab, being the transverse axes.

Cor. 3. When F coincides with A, then SQ=SA; and if the ellipse anbd becomes a circle whose radius=SA=Sq, and therefore l=2SA; we have FG : fg :: $\frac{\sqrt{L}}{SA} : \frac{\sqrt{2SA}}{SA}$ :: $\sqrt{L}$: $\sqrt{2SA}$; that is, vel. at F : vel. at f :: $\sqrt{L}$: $\sqrt{2SA}$.

Cor. 4. In the same manner, when F coincides with B, and the ellipse anbd becomes a circle, we have vel. at F : vel. at f :: $\sqrt{L}$: $\sqrt{2SB}$.

Cor. 5. When F coincides with N, then SF=SN=AC, and SQ=CN. And if the ellipse anbd, becomes a circle whose radius=SN=AC, therefore l=2AC, and Sq=AC, the general proportion becomes FG : fg :: $\frac{\sqrt{L}}{CN} : \frac{\sqrt{2AC}}{AC}$; ther. $FG^2 : fg^2 :: \frac{L}{CN^2} : \frac{2AC}{AC^2} :: \frac{2}{AC} : \frac{2}{AC}$ ∴ FG=fg.

Therefore the velocities of bodies revolving in ellipses at the mean distances, are equal to those in circles at the same distances.

Cor. 6. If the ellipse anbd becomes a circle, whose radius=SF=Sf=Sq; and ther. l=2SF; we then have $FG^2 : fg^2 :: \frac{L}{SQ^2} : \frac{2SF}{SF^2} = \frac{2}{SF}$; ther. $FG^2 : fg^2 :: SF \times L : 2 \times SQ^2 :: SF \times \frac{1}{2}L : SQ^2$; or, FG : fg :: $\sqrt{SF \times \frac{1}{2}L}$: SQ :: vel. in ellip. : vel. in circle when the distance SF=radius of the circle.

Prop. VIII. Fig. 1.

Let the ellipses AGD, agd, be described about the common focus S, with forces which vary inversely as the squares of the distances from the focus; then angular vel. at F : angular vel. at f :: $\frac{SFG}{SF^2} : \frac{Sfg}{Sf^2}$.

Let FG, fg, be described in indefinitely little equal parts of time; join SF, SG, Sf, Sg; and we then have SF×SG×sin. FSG : Sf×Sg×sin. fSg :: SFG : Sfg;

Ther. sin. FSG : sin. fSg :: $\frac{SFG}{SF\times SG} : \frac{Sfg}{Sf\times Sg}$;

That is, ang. vel. at F : ang. vel. at f :: $\frac{SFG}{SF^2} : \frac{Sfg}{Sf^2}$ :: $\frac{1}{SF^2} : \frac{1}{Sf^2}$ in the same orbit; for then SFG, Sfg would be equal.

Cor. 1. Ang. vel. at F : ang. vel. at f :: $\frac{\sqrt{L}}{SF^2} : \frac{\sqrt{l}}{Sf^2}$ (Prop. V.)

Cor. 2. The angular velocity in an ellipse at a mean distance, is to that in a circle at the same distance, as the conjugate axis to the transverse axis.

When F coincides with N, then SF=SN=AC, and SQ=CN; and if the ellipse a n b d becomes a circle whose radius=SN=AC=Sf, then l=2AC.

Let φ=angular velocity in the ellipse, and θ=angular velocity in the circle; then the proportion in cor. 1. will become,

$\varphi : \theta :: \frac{\sqrt{L}}{AC^2} : \frac{\sqrt{2AC}}{AC^2} :: \sqrt{L} : \sqrt{2AC}$;

Ther. $\varphi^2 : \theta^2 :: L : 2AC :: \frac{2CN^2}{AC} : 2AC :: 2CN^2 : 2AC^2$;

That is, $\varphi : \theta$ :: CN : AC :: ang. vel. in ellip. : ang. vel. in circle.

Cor. 3. Since $\varphi : \theta :: \frac{\sqrt{L}}{SF^2} : \frac{\sqrt{2AC}}{AC^2}$; then if $\varphi = \theta$, $\frac{\sqrt{L}}{SF^2} = \frac{\sqrt{2AC}}{AC^2}$ $\therefore SF^2 = \frac{AC^2\sqrt{L}}{\sqrt{2AC}} = \frac{AC^2}{\sqrt{AC}\times\sqrt{2}} \times \frac{CD\times\sqrt{2}}{\sqrt{AC}} =$ AC×CD; ther. $SF = \sqrt{AC\times CD}$.

Hence the distance in the ellipse may be found, at which the angular velocity is equal to that in the circle at the same distance.

Prop. IX. Fig. 1.

Let AGD be an ellipse, described by a body acted upon by a centripetal force tending to the focus S.

The centrifugal force at F : centrifugal force at f :: $\frac{1}{SF^3} : \frac{1}{Sf^3}$.

Suppose the body in the same indefinitely little part of time to describe the arcs FG, fg; with the center S, and distances SG, Sg, describe the circular arcs Gm, gv; draw GL perpendicular to SF, and gl perpendicular to Sf; then Lm will represent the centrifugal force at F, and lv the centrifugal force at f.

Because FG, fg, are indefinitely little; $SF^2 \times GL^2 = Sf^2 \times gl^2$; and $Lm : lv :: \frac{GL^2}{SL} : \frac{gl^2}{SL} :: \frac{GL^2}{SF} : \frac{gl^2}{Sf}$, ultimately; but $\frac{GL^2}{SF} = \frac{Sf^2 \times gl^2}{SF^3}$, and $\frac{gl^2}{Sf} = \frac{SF^2 \times GL^2}{Sf^3}$; therefore $Lm : lv :: \frac{Sf^2 \times gl^2}{SF^3} : \frac{SF^2 \times GL^2}{Sf^3} :: \frac{1}{SF^3} : \frac{1}{Sf^3}$.

Ther. centrif. force at F : centrif. force at f : $\frac{1}{SF^3} : \frac{1}{Sf^3}$.

Prop. X. Fig. I.

Let AGD be an ellipse described by a body acted upon by a centripetal force tending to the focus S; then centripetal force at F : centrifugal force at F :: $2SF^3 : SQ^2 \times FE$.

Let FH be the diameter of the circle of curvature at F; FE, the chord of curvature passing through the focus S; draw SQ perpendicular to the tangent at F. and GL perpendicular to SF; draw Gz parallel to FP; with the center S, and distance SG, describe the circular arc Gm, and join HE.

In the similar triangles GLz, SFQ, we have $Gz^2 : GL^2 :: SF^2 : SQ^2$. But centrip. force at F : centrif. at F :: Fz : Lm; But (ultimately) $Fz = \frac{Gz^2}{FE}$, and $Lm = \frac{GL^2}{2SG} = \frac{GL^2}{2SF}$; therefore $Fz : Lm :: \frac{SF^2}{FE} : \frac{SQ^2}{2SF} :: 2SF^3 : SQ^2 \times FE$ :: Centrip. force at F : centrif. force at F.

Cor. 1. By 31 Ellip. $SQ^2 = \frac{SF^2 \times CD^2}{CI^2}$, and (3 cir. cur.)

$FE = \frac{2CI^2}{AC}$; ther. $SQ^2 \times FE = \frac{2SF^2 \times CD^2}{AC}$; hence we have centrip. force at F : centrif. force at F :: $2SF^2 : \frac{2SF^2 \times CD^2}{AC}$ :: $SF : \frac{CD^2}{AC}$ :: SF : $\frac{1}{2}$ latus-rectum.

COROLLARY 2.

If the centripetal force $GP = Fz$, at any point F, be put $= M$, and the centrifugal force at the same point, be put $= N$, the principal latus-rectum being $tT = XY$ - - - - - - - - $= L$,

We shall then have the following proportions:

$M : N ::$ When F is at

- $SF : \frac{CD^2}{AC} :: SF : \frac{1}{2}L$ -- Any point in the Ellipse,
- *$SA : \frac{CD^2}{AC} :: CB : BS$ -- the point A,
- †$SB : \frac{CD^2}{AC} :: CB : AS$ -- the point B,
- $SN : \frac{CD^2}{AC} :: CB^2 : CD^2$ -- D or N,
- $St : \frac{CD^2}{AC} :: \frac{1}{2}L : \frac{1}{2}L$, Therefore - - - $M = N$ -- T or t,
- ‡$SX : \frac{CD^2}{AC} :: AB - \frac{1}{2}L : \frac{1}{2}L$ -- X or Y.

SCHOLIUM.

Hence it appears that if a body revolve in an elliptic orbit AGD, about the focus S, and is acted upon by a centripetal force, which varies inversely as the squares of the distances from the focus, during the time the body is describing the space

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* $SA : \frac{CD^2}{AC} :: AC + CS : \frac{(AC+CS)(AC-CS)}{AC} :: AC : AC - CS :: CB : BS.$

† $SB : \frac{CD^2}{AC} :: AC - CS : \frac{(AC+CS)(AC-CS)}{AC} :: AC : AC + CS :: CB : AS.$

‡ $SX + XM = 2AC$ ∴ $SX = 2AC - MX = AB - \frac{1}{2}L.$

TDt, the centripetal force will exceed the centrifugal force: and while it is describing the space tBT, the centrifugal force will exceed the centripetal force: and at the points t and T, the both forces will be equal.

Prop. XI. Fig. 1.

If a body describe an ellipse AGD, about the focus S, with forces which vary inversely as the squares of the distances from the focus; the velocity in the ellipse is to that of a body revolving in a circle at the same distance from the center, in a less ratio, than $\sqrt{2}$ to 1.

Let AB, DN, be the transverse and conjugate diameters; S, M, the foci, and C the center; let FG be described in an indefinitely little part of time; join SF, SG, SN, MF; draw GP parallel to SF, and CI parallel to the tangent FP; let FEH be the circle of curvature at F, and FE the chord passing through the focus S.

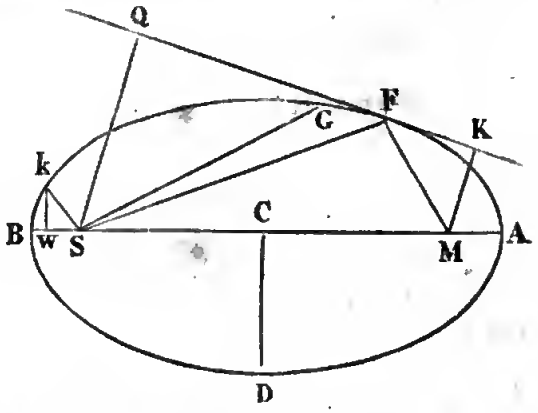
Then (1. cir. cur.) vel. $\propto \sqrt{PG \times FE} \propto \sqrt{PG \times \frac{2CI^2}{AC}}$, (3. cir. cur.); in a circle CI=CA; in that case, vel. $\propto \sqrt{PG \times 2AC}$; ther. vel. in the ellipse at F : vel. in a circle at the same distance :: $\sqrt{PG \times \frac{2CI^2}{AC}} : \sqrt{PG \times 2AC} :: \sqrt{\frac{2CI^2}{AC}} : \sqrt{2SF}$ (in the circle AC=SF) :: $\sqrt{\frac{2SF \times FM}{AC}} : \sqrt{2SF} :: \sqrt{FM} : \sqrt{AC} :: \sqrt{2AC-SF} : \sqrt{AC}$; by 26 Ellip. $SF \times MF = CI^2$, and 10 Ellip. $FM = 2AC - SF$. Therefore the velocity in the Ellipse at F, is to the velocity in a circle at the same distance, as $\sqrt{2AC-SF}$ to $\sqrt{AC}$, which is always less than $\sqrt{2}$ to 1.

Cor. When $SF = SN = AC$ (6 Ellip.), then the velocity in the ellipse in F is equal to the velocity in the circle.

Prop. XII.

Supposing a body acted upon by a centripetal force tending to the focus S, describe the ellipse BFAD, to be projected from

a given point F, in a given direction F Q with a given velocity, to find the axes of the ellipse and the periodic time.



Let A B be the greater axis, C D half the lesser, and S, M, the foci; suppose F G to be an indefinitely small part of the curve described in the particle of time t, and f a space, a body will descend at F in the time t by the centripetal force; join S G, and make the area S B k equal to S F G; demit the perpendiculars S Q, M K, k w, and put s for the nat. sin. (rad. 1.) of the angle S F Q, or M F K; we shall then have

$SQ^2 : SB^2 :: Bk^2 : FG^2$ from the equal triangles,
$SB^2 : SF^2 :: f : Bw$ proposition 2,
$AB : 4CD^2 :: Bw : Bk^2$ proposition 5 cir. cur.
$4CD^2 : 4SQ :: MK : 1$ proposition 26 ellipse.

From whence $AB.SQ : 4SF^2 :: f.MK : FG^2$.

Therefore $AB = \frac{4f.MK.SF^2}{SQ.FG^2} = \frac{4f.SF(AB-SF)}{FG^2} = \frac{4f.SF^2}{4f.SF-FG^2}$, and $CD^2 = s^2.SF(AB-SF) = \frac{s^2.SF^2.FG^2}{4f.SF-FG^2}$

$\therefore CD = \frac{s.SF.FG}{(4f.SF-FG^2)^{\frac{1}{2}}}$, and $FM = AB-SF = \frac{SF.FG^2}{4f.SF-FG^2}$.

Now since F M and the angle M F K are given, the focus M is given; therefore the position of the transverse axis A B is given.

To find the Periodic Time.

By putting p for the circumference of a circle whose diameter is unity, the area of the ellipse will be $AB.2CD.\frac{p}{4} = \frac{2psf.FG.SF^2}{(4f.SF-FG^2)^{\frac{3}{2}}}$. Then. def. 4. $\frac{s.SF.FG}{2} : t :: \frac{2psf.FG.SF^2}{(4f.SF-FG^2)^{\frac{3}{2}}} : \frac{4pft.SF^2}{(4f.SF-FG^2)^{\frac{3}{2}}}$ time of one revolution.

Cor. 1. $\frac{4CD^2}{AB} = \frac{s^2.FG^2}{f}$ = latus-rectum of the transverse axis AB.

Cor. 2. The transverse axis and periodic time will remain the same, whatever be the angle of direction SPQ; for their values are not affected with s.

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Prop. I. Fig. 1.

Let A B C be any curve whose axis is A D; and let E F G be a curve line, so related to A B C, that if from any point H in A D there be drawn H B perpendicular to A D, meeting the curves A B C, E F G, in B and F, and if from B there be drawn the tangent B f, meeting A D in f, so that $fH \times HF : HB^2 :: p : q$, a given ratio; and if A E be drawn perpendicular to A D; then will $fH \times HF = 2 \times$ area A E F H.

Let A H be divided into indefinitely little parts, in the points K, L, M, &c.; draw K N, L O, M P, &c. parallel to H B, meeting the curves in N, O, P, Q, R, S, &c.; draw B T parallel to A D; and let $HB \times BT = fH \times HF$, and join H T; draw N V, O X, P Y, &c. parallel to A D; and let N Z be a tangent to the curve at N; let N V meet B H in a, and O X meet B H, N K, in b, c.

Because $HB \times BT = fH \times HF$, $BT : HF :: fH : HB ::$ N a (K H) : B a, (the triangles f H B, N a B, being considered as similar); ther. $aB \times BT = KH \times HF$. Again, because $ZK \times KQ : KN^2$ (H a^2) :: P : q :: $fH \times HF : HB^2$, and $Ha^2 : Ha \times aV :: HB^2 : HB \times BT$; ther. $ZK \times KQ : Ha \times aV :: fH \times HF : HB \times BT$; therefore $ZK \times KQ = Ha \times aV$ ∴ V a : K Q :: Z K : H a (K N) :: o c (L K) : C N (a b); ther. $Va \times ab = KQ \times LK$.

Again let P Y meet B H in d, and in the same way it is shown that $db \times bX = ML \times LR$, and so on; ther. the sum of the rectangles a B T, b a V, d b X, &c. is equal to the sum of the

rectangles KHF, LKQ, MLR, &c. But the former sum may be considered equal to the triangle HBT, and the latter sum to the figure AEFH: but fH×HF=HB×BT=double the triangle HBT; ther. fH × HF = 2 × area AEFH. Q.E.D. From the general proportion we have HF : KQ :: $\frac{HB^2}{fH}$: $\frac{KN^2}{ZK}$; that is, HF, KQ, &c. are as the squares of the ordinates directly, and subtangents inversely.

Prop. II. Fig. 2.

Let there be a curve ABC, whose axis is AD; and let EFG be a curve, whose asymptote is AH, perpendicular to AD; let the curves be so related to each other, that if from any points B, C, in ABC, there be drawn BK, CD, perpendicular to AD, meeting the curve EFG in F, G; and let BL, CM, be tangents to ABC at the points B and C; and if MD×DG : LK×KF :: CD : BK; the figure HADGE=MD×DG.

Draw FN parallel to BL; let CD be divided into indefinitely little parts in O, P, Q, R, S, &c.; draw OT, PV, QX, RY, SZ, &c. parallel to AD; and draw Ta, Vb, Xc, Yd, Ze. &c. parallel to CD, meeting the curve EFG in f, g, h, k, l, &c.; draw the tangents Tm, Vo; and let VP meet Ta in p.

Because FN is parallel to BL, BK : KL :: FK : KN; ther. BK×KN=LK×KF; but (hypoth.) MD×DG : LK×KF :: CD : BK :: CD×KN : BK×KN ∴ MD×DG=CD×KN; ther. DG : KN :: CD : MD :: CO : OT (Da) ∴ GD×Da=CO×KN.

Again, because ma×af : LK×KF :: aT : BK :: aT×KN : BK×KN ∴ ma×af=aT×KN; ther. af : KN :: Ta : am :: Tp : pV (ab) ∴ af×ab=KN×Tp=KN×OP. In the same way it may be shewn that gb×bc=PQ×KN, and so on; therefore GDa+fab+gbc+&c.=CO×KN+OP×KN+PQ×KN+&c.=(CO+OP+PQ+&c.)×KN=CD×KN=MD×DG; but the sum of the rectangles GDa, fab, gbc, &c. may be considered equal to the figure HADGE, as

well as CO+OP+PQ+&c.=CD; ther. HADGE=MD× DG. Q.E.D.

Prop. III. Fig. 3.

Let a body descend in the line AB from A, supposing any centripetal force whatever: let CDE be a curve, such, that if from any two points FG, FD, GE, be drawn perpendicular to AB, so that centripetal force at F : centripetal force at G :: FD : GE, and AC be drawn perpendicular to AB; then will velocity at F : velocity at G :: $\sqrt{ACDF}$: $\sqrt{ACEG}$.

Let AHK be a curve, such, that if from any points F, G, there be drawn FH, GK, perpendicular to AB, so that FH : GK :: vel. at F : vel. at G, and HL, KM, be tangents to the curve in H, and K; let FN, GO, be spaces passed over, in an indefinitely little part of time; draw NP, OQ, perpendicular to AB; and HR, KS, parallel to AB; and let KG×GT=FH× GE, or GK : FH :: GE : GT.

Because FH : NP :: vel. at F : vel. at N,
And - - - - GK : OQ :: vel. at G : vel. at O,
Also - - - - PR : QS :: incr. of vel. at F : incr. vel. at G,

The velocities being as the forces and times, and the times being equal, the velocities are therefore as the forces: therefore PR : QS :: centrip. force at F : centrip. force at G :: FD : GE :: FD×FH : GE×FH=KG×GT; and because the spaces FN, GO, are passed over in the same time, and the spaces being as the velocities and times;

Ther. vel. F : vel. at G :: FN : GO :: HR : KS.
But HR : RP :: LF : FH :: LF×FD : FH×FD;
And RP : QS :: FD×FH : KG×GT;
Ther. HR : QS :: LF×FD : KG×GT;
But QS : SK :: KG : GM :: KG×GT : GM×GT;
Ther. HR : SK :: LF×FD : GM×GT;
But vel. at F : vel. at G :: HR : KS :: FH : GK (hypoth.)
Ther. FH : GK :: LF×FD : GM×GT :: FH^2 : FH× GK :: FH×GK : GK^2;

But (hypoth.) FH : GK :: GT : GE :: GT×MG : GE× MG :: FH×GK : GK²;

Ther. LF×FD : MG×GE :: FH² : GK²; ther. (Prop. I), LF×FD=2×area ACDF, and MG×GE=2×ACGE.

Ther. LF×FD : MG×GE :: ACDF : ACGE :: FH² : GK²;

Ther. FH : GK :: $\sqrt{ACDF}$: $\sqrt{ACGE}$ :: velocity at F : velocity at G. Q.E.D.

Prop. IV. Fig. 4.

Let a body descend from A, in the line AB with any centripetal force, and let CDE be a curve, such, that if from any two points F, G, FD, GE, be drawn perpendicular to AB, and that GE : FD :: vel. at F : vel. at G; and let AH, perpendicular to AB, be the asymptote of the curve CDE; then will the

Time of descent through AF : time of descent through AG :: HAFDC : HAGEC.

Let AKL be a curve, such, that drawing FK, GL, perpendicular to AB, so that

FK : GL :: time of descent through AF : time of descent through AG, draw the tangents KM, LN; and let FO, GP, be the spaces passed over by the body in an indefinitely little part of time, when at F and G; draw OQ, PR, perpendicular, and KS, LT, parallel to AB.

Because PR : GL :: time descent through AP : time descent through AG,

And OQ : FK :: time descent through AO : time descent through AF,

Then RT : QS :: time descent through GP : time descent through FO,

But, the time of descent through GP=the time of descent through FO,

Therefore RT=QS. Because the spaces GP, FO, are passed over in equal times.

The vel. at G : vel. at F :: GP : FO :: LT : KS,

Since, MF×FD : MF×EG :: FD : EG :: LT : KS,

And MF×EG : FK×EG :: MF : FK :: KS : SQ,

Ther. MF×FD : FK×EG :: LT : SQ (RT) :: NG : GL :: NG×GE : GL×GE.

And MF×FD : NG×GE :: FK×GE : GL×GE :: FK : GL.

Therefore (Prop. II.) HAFDC=MF×FD, and HAGEC =NG×GE;

Hence FK : GL :: HAFDC : HAGEC;

Ther. time of descent through AF : time of descent through AG :: HAFDC ; HAGEC. Q.E.D.

Prop. V. Fig. 5.

Suppose a body to descend from A, in the line AB, and ACDE to be a curve so related to AB, that if from any two points F, G, there be drawn FC, DG, perpendicular to AB, and that the vel. at F : vel. at G :: $\sqrt{FC}$: $\sqrt{GD}$, and the tangents CH, DK, be drawn; then will the centripetal force at F : centripetal force at G :: $\frac{FC}{FH} : \frac{GD}{GK}$.

Let LMN be a curve so related to AB, that CF, DG, being produced to M and N, the centripetal force at F : centripetal force at G :: FM : GN; take FO, indefinitely little, draw OP parallel to FC; take GQ equal to FO, and draw QR parallel to GD; draw CS, DT, parallel to AB; produce PO and RQ, to meet the curve LMN, in V and X; and draw AL perpendicular to AB.

Because vel. at F : vel. at G :: $\sqrt{FC}$: $\sqrt{GD}$ :: $\sqrt{LAFM}$: $\sqrt{LAGN}$ (Prop. III.)

Ther. LAFM : LAGN :: FC : GD; which proportion shows the relation of the curves ACD, LMN, to each other.

Ther. LAOV : LAFM :: OP : FC,

And, MF×FO : LAFM :: OP−FC : FC :: SP : FC,

But, LAFM : LAGN :: FC : GD,

Ther. MF×FO : LAGN :: SP : GD,

Also, LAGN : LAQX :: GD : QR,

Ther. L A G N : N G × G Q :: G D : Q R − G D = R T,

And, M F × F O : N G × G Q :: S P : R T, but F O = G Q (hypoth.)

Ther. F M : G N :: M F × F O : N G × G Q :: S P : R T,

Because F O and G Q, are indefinitely little, the triangles H F C, C S P, may be considered as similar, as well as the triangles K G D and D T R.

Ther. H F : F C :: C S (F O) : S P ∴ P S × H F = C F × F O,

In the same way it may be shewn, that R T × G K = D G × G Q,

Ther. P S × H F : R T × G K :: C F × F O : D G × G Q :: F C : D G :: C F × G K : D G × G K,

Hence P S × H F : C F × G K :: R T × G K : D G × G K :: R T : D G :: R T × H F : D G × H F,

Ther. P S × H F : R T × H F :: C F × G K : D G × H F :: P S : R T :: F M : G N,

Hence, F M : G N :: C F × G K : D G × H F,

But hypoth. centrip. force at F : centrip. force at G :: F M : G N,

Ther. centrip. force at F : centrip. force at G :: F C × G K : G D × F H :: $\frac{FC}{FH} : \frac{GD}{GK}$. Q. E. D.

Prop. VI. Fig. 6.

If a body acted upon by a centripetal force, tending to a fixed center, describe a curve line, the angular velocity of the body round the fixed center will be reciprocally as the square of the distance of the body from the center.

Let A B C be any curve described by a body that is acted upon by a centripetal force, tending to the center S; draw S A, S B, to any two points A, B, in the curve; the angular vel. of S A : angular vel. of S B :: $SB^2 : SA^2 :: \frac{1}{SA^2} : \frac{1}{SB^2}$.

Let A D, B E, be described in indefinitely little parts of time; join S D, S E : with the center S, and radii S D, S E, describe the arcs F H, E K, meeting S A, S B, S E, in F, G, H, K.

Because A D, B E, are described in equal times, the triangles A S D, B S E, will be equal.

Ther. sin. A S D : sin: B S E :: S B × S E : S A × S D,

That is, D F : G H :: $SB^2 : SA^2 :: \frac{1}{SA^2} : \frac{1}{SB^2}$,

A D, B E, being indefinitely small. Q. E. D.

Prop. VII. Fig. 7.

Let S A, S B, be two right lines intersecting each other in the point S; and A C, B D, perpendicular thereto; and let A E, B F, be indefinitely little parts described by bodies moving therein in an indefinitely little part of time; and let $BF^2 : AE^2 :: SB : SG$,

Then, centrifugal force at B : centrifugal force at A :: S A : S G;

That is, F L : E H :: S A : S G,

Join S E, S F; with the center S, and radii S A, S B, describe circles meeting S E in H and K, and S F in L and M. Because $BF^2 = FM \times FL$, and $AE^2 = EK \times EH$, $BF^2 : AE^2 :: FM \times FL : EK \times EH$.

But B F, A E, being indefinitely small, we may consider M L = M F, and K E = K H,

Ther. $BF^2 : AE^2 :: LM \times FL : KH \times EH :: SL \times FL : SH \times EH$;

But, $BF^2 : AE^2 :: SB(SL) : SG :: SL \times LF : SG \times LF$,

Ther. $SL \times FL : SH \times EH :: SL \times LF : SG \times LF$,

Hence $SH \times EH = SG \times LF$ ∴ F L : E H :: S A : S G.

But F L = centrifugal force at B,

And E H = centrifugal force at A,

Ther. centrifugal force at B : centrifugal force at A :: S A : S G. Q. E. D.

Prop. VIII. Fig. 8.

Let A B C be any curve, described by a body that is acted upon by a centripetal force, tending to the center S: the centri-

fugal force at B : centrifugal force at A :: $SA^3 : SB^3 :: \frac{1}{SB^3} : \frac{1}{SA^3}$.

Suppose a body in the same indefinitely little part of time to describe the arcs AD, BE; draw DK, EL, perpendicular to SA, SB; complete the parallelograms AKFD, BLEG, and let $BG^2 : FA^2 :: SB : SH$; join SD, SE, SF, SG. Because the arcs AD, BE, are indefinitely little, they may be considered as straight lines; the force acting in the direction AD, may be resolved into the forces AF, AK; likewise the force acting in the direction BE, may be resolved into the forces BG, BL. Because AD, BE, are described in equal times, the triangles SDA, SFA; SBE, SBG, will be equal to each other.

Ther. $SA^2 : SB^2 :: BG^2 : AF^2 :: SB : SH \therefore SA^2 \times SH = SB^3$.

Ther. $SA^3 : SB^3 :: SA^2 : SA^2 \times SH :: SA : SH$.

But (Prop. VII.) centrifugal force at B :: centrifugal force at A :: SA : SH.

Ther. centrifugal force at B : centrifugal force at A :: $SA^3 : SB^3 :: \frac{1}{SB^3} : \frac{1}{SA^3}$. Q. E. D.

Prop. IX. Fig. 9.

Supposing a body acted upon by a centripetal force tending to the center S, to be projected at the point A, in the direction AB, perpendicular to SA; and suppose in an indefinitely little part of time, the body would describe AC by the projectile force alone; let AD be the space that would be described in the same time by the centripetal force tending to the center S; then,

Centrifugal force at A : centripetal force at A :: $AC^2 : 2SA \times AD$.

Join SC, and with the center S, and distance SA, describe a circle meeting SC in E and F.

Because $CE : AD :: CE \times FC : AD \times FC :: AC^2 : AD \times FC$, and CE being infinitely little, we may consider FC=FE; then $AD \times FC = AD \times FE = 2SA \times AD$.

Ther. $CE : AD :: AC^2 : 2SA \times AD$.

And, centrifugal force at A : centripetal force at A :: $AC^2 : 2SA \times AD$. Q. E. D.

Prop. X. Fig. 10.

Supposing a body, in an indefinitely little part of time, to describe the part A D with an uniform velocity, and a body acted upon by a centripetal force tending to the center S, to describe the part A E, in the same time: then

Vel. at E : vel. in A D :: 2 A E : A D.

Because the time in which A E is described is indefinitely little, the centripetal force may be considered as acting uniformly during such little part of time, therefore the body, with the velocity acquired at E, acting uniformly, would describe twice A E in the same time that A E was described, that is, in the same time that A D was described with an uniform velocity; therefore, vel. at E : vel. in A D :: 2 A E : A D. Q. E. D.

Prop. XI. Fig. 11.

Supposing a body acted upon by a centripetal force tending to the center S, to descend in the straight line A S from the point A; let there be a curve C D E, such, that if from any two points A, B, there be drawn A C, B D perpendicular to A S, so that the centripetal force at A : centripetal force at B :: A C : B D; also let the body at the point B be projected in the direction B F, perpendicular to S B, with the velocity that would be acquired in falling from A to B.

The centrifugal force at B, arising from the projectile force in the direction B F - - - - - - - - - - - - } :

Centripetal force at B, tending to the center S - - - - - :: 2 × space C A B D : S B × B D.

Let the body, in an indefinitely little part of time, descend from B to H by the centripetal force; and in an equal time, let the body, by the projectile force in the direction B F, describe B G; draw H K parallel to B D, and let D B × B L = C A B D.

Because (Prop. III.) vel. at B, by falling from A to B, :
vel. at H, by falling from B to H, ::
$\sqrt{CABD} : \sqrt{DBHK}$.

and the space CABD = DB×BL, and DB×BH = DBHK, because BH is indefinitely little;

Ther. vel. at B : vel. at H :: $\sqrt{DB \times BL}$: $\sqrt{DB \times BH}$: $\sqrt{LB}$: $\sqrt{BH}$.

But the vel. at B, by falling from A to B, is equal to the velocity in BF; ther. vel. in BF : vel. at H :: $\sqrt{LB}$: $\sqrt{BH}$;

But (Prop. X.) vel. in BF : vel. at H :: BG : 2BH; ther. BG : 2BH :: $\sqrt{LB}$: $\sqrt{BH}$ ∴ $BG^2 : 4 \times BH^2$ :: LB : BH :: 4BH×LB : $4BH^2$ ∴ $BG^2 = 4LB \times BH$; but (Prop. IX.) centrifugal force at B : centripetal force at B :: BG^2 : 2SB×BH :: 4LB×BH : 2SB×BH :: 2LB : SB :: 2LB×BD : SB×BD :: 2×CABD : SB×BD. Q.E.D.

Corollary.

In SB take any point a, and draw ac parallel to BD; let the body at the point B be projected in the direction BF, perpendicular to SB, with a velocity that would be acquired in falling from B to a;

The centrifugal force at B arising from the projectile force in direction BF - - - - - - - - - - - - - - } :

Centripetal force at B, tending to the center S - - - - - : : 2×caBD : SB×BD.

This may be demonstrated in the same way as the proposition. Or it may be shewn that the centrifugal force at a : centripetal force at a : : 2×caBD : Sa×ac.

Prop. XII. Fig. 12.

Let a body acted upon by a centripetal force tending to the center S, describe the curve ABC; and let AD, BE, be tangents to the curve at any points A, B, in the curve; let SD, SE, be perpendicular to AD, BE.

The vel. at A : vel. at B : : SE : SD.

Join SA, SB, and let AF, BG, be parts of the curb described in indefinitely little equal parts of time; join SF, SG, because AF, BG, are described in equal times, the triangles ASF, BSG

will be equal; therefore AF : BG :: SE : SD; but AF : BG :: vel. at A : vel. at B. Therefore the vel. at A : vel. at B :: SE : SD. Q. E. D.

Prop. XIII. Fig. 13.

If a body acted upon by any centripetal force, be any way moved, and another body descend or ascend in a right line, and their velocities be equal in any one case of equal altitudes, their velocities will be also equal at all equal altitudes.

Let a body descend from A, through D and E to the center C; and let another body move from V, in the curve VBk; from the center C, with any distances infinitely near to each other, describe the circles DB, EK; draw CB, meeting KE in N; on BK let fall the perpendicular NT; and imagine the bodies at D and B to have equal velocities: then, because the distances CD, CB, are equal, the centripetal forces in D and B will be equal. (Prop. VIII.) Let these forces be expressed by the equal lines DE, BN; and let the force BN be resolved into the forces NT and TB; the force NT will not affect the velocity of the body in the curve; but the other force BT will be employed in accelerating it. Because B and K, and likewise D and E, are indefinitely near to each other; the accelerating force from B to K, and from D to E, may be considered uniform; therefore the accelerations of the bodies in D and B, produced in equal times, are as the lines DE, BT;

Ther. vel. at E : vel. at T :: DE (BN) : BT;

But because BNK is a right angle, $BN = \sqrt{BT} \times \sqrt{BK}$;

Ther. vel. at E : vel. at T :: $\sqrt{BT} \times \sqrt{BK}$: BT :: $\sqrt{BK}$: $\sqrt{BT}$;

But, vel. at K : vel. at T :: $\sqrt{BK}$: $\sqrt{BT}$, (the spaces being as the squares of the velocities);

Ther. vel. at E : vel. at T :: vel. at K : vel. at T;

Therefore the velocity at E is equal to the velocity at K. Q. E. D.

Prop. XIV. Fig. 14.

Let there be a semicircle whose diameter is AB, and center C; bisect CB in D; draw DE perpendicular to AB, and join AE. If from any point F in the semicircle there be drawn FG, perpendicular to AB, meeting AB in G, and AE in H, the DE^2 will be greater than FG×GH.

Let EK, a tangent at E, meet AB in K, and FG in L, and join CE; then by a property of the circle, CK : CE :: CE : CD, or CK : CB :: CB : CD; because CB=2CD ∴ CK=2CB ∴ CB=BK=CA;

And because BD=DC ∴ AD=DK.

And because AD^2 : AD×AG :: AD : AG :: DE : GH :: DE^2 : DE×GH;

Ther. AD^2 : DE^2 :: AD×AG : DE×GH.

Since AG×GK : AG×DA :: GK : AD(DK) :: GL : DE :: GL×GH : DE×GH; ther. AG×GK : LG×GH :: AG×AD : DE×GH;

Hence AD^2 : DE^2 :: AG×GK : LG×GH.

But AD^2 is greater than AG×GK; ther. DE^2 is greater than LG×GH; but GL is greater than GF, therefore DE^2 is greater than FG×GH. Q. E. D.

Prop. XV. Fig. 15.

In the straight line AB, take BC, so that AC may be triple CB; and let D be any point in AB; the ratio of the cube of DA to the cube of AC will be less than the ratio of BC to BD.

Upon AB describe a semicircle AFB; and let CF be perpendicular to AB; join AF, and draw DG parallel to CF, meeting AF in H; and let KD×DH=CF^2.

Because KD×DH=CF^2; DH : CF :: CF : DK;

But DH : CF :: AD : AC :: CF : DK;

Ther. AD^2 : AC^2 :: CF^2 : DK^2 :: AD^3 : AC^3×AD;

But, AC^2×AD : AC^3 :: AD : AC :: AD×CB : AC×CB=CF^2;

Ther. AD^3 : AC^3 :: AD×CB : DK^2;

But (Prop. XIV.) CF^2 is greater than GD×DH;

Therefore KD×DH is greater than GD×DH;

Hence KD is greater than GD; or KD^2 greater than GD^2;

But $GD^2 = AD \times DB$; ther. KD^2 is greater than AD×DB;

Hence the ratio of AD×BC to AD×BD, or of BC to BD, is less than the ratio of AD×BC to DK^2;

Ther. the ratio of AD^2 to AC^2 is less than the ratio of BC to BD.

Prop. XVI. Fig. 16.

Let A be a given point in the straight line AB; and CDE a curve, such, that if from any two points C, D, there be drawn CF, DG, perpendicular to AB, so that $AG^3 : AF^3 :: CF : DG$, and that $AH = 4HF$, and CH be joined; then will CH be a tangent to the curve in the point C.

From any point K, in CH, draw KG perpendicular to AB, meeting AB in G, and the curve in D. Because $AH = 4HF$. $\therefore HF = \frac{1}{4}AH$; and $AH - HF = AF = AH - \frac{1}{4}AH = \frac{3}{4}AH$; ther. AF : HF :: 3 : 1 $\therefore AF = 3HF$; hence (Prop. XV.) $AG^3 : AF^3$ is less than HF : HG; but (hypoth.) $AG^3 : AF^3 :: CF : DG$; and FH : HG :: FC : GK, (by sim. triangles); therefore CF : DG is less than FC : GK; therefore DG is greater than GK; therefore the point K is without the curve CDE. In the same way it is shewn, that any other point in CH is without the curve CDE; therefore CH is a tangent to the curve CDE at the point C. Q. E. D.

Prop. XVII. Fig. 17.

Let A be a given point in the straight line AB, and CDE a curve, such, that if from any two points D, E, in the curve, there be drawn DF, EG, perpendicular to AB, $DF : EG :: AG^3 : AF^3$;—also draw AH perpendicular, and DK, EL, parallel to AB; then will the space DKLE = 3×DFGE, and AF×FD − AG×GE = 2×DFGE.

Suppose KL to be divided into indefinitely little parts in M, N, &c.; draw MO, NP, &c. parallel, and OV, PX, &c. perpendicular to AB; let DF, OV, PX, meet MO, NP, LE, in

q, r, s; and let T be a point in A B, such, that A T=4 F T, and join D T.

Because (Prop. XVI.) $AG^3 : AF^3 :: DF : EG$, and A T = 4 F T; D T will be a tangent to the curve at D; and because M O is indefinitely near to D K, the triangles D F T, D q O, may be considered as similar; ther. D F : F T :: D q : q O, ∴ F T × D q = D F × q O = F O; but because A T = 4 F T, A F = 3 F T ∴ A F × D q = 3 F T × D q ∴ D M = 3 F O. The same way it may be shewn that O N = 3 V P, P L = 3 E X, &c. But D M + O N + P L + &c. = D K L E, and F O + V P + E X + &c. = D F G E; ther. 3 F O + 3 V P + 3 E X + &c. = D K L E, therefore the space D K L E = 3 × D F G E.

Again let D F meet E L in Y, because D K L E = 3 × D F G E ∴ D L + D Y E = 3 D Y E + 3 F E, or D L = 3 F E + 2 D Y E = F E + 2 F E + 2 D Y E = F E + 2 × D F G E; add L F to both sides of the equation, then D L + L F = F E + 2 × D F G E + L F ∴ A D = A E + 2 × D F G E, ther. A F × F D − A G × G E = 2 × D F G E. Q. E. D.

Prop. XVIII. Fig. 18.

Let S be a given point in the straight line A S, and C D E a curve, such, that if from any two points A, B, there be drawn A C, B D, perpendicular to A S,

$BD : AC :: AS^3 : SB^3$; and let S A = S F. Then,

S A × A C : C A B D :: $2SB^3 : AB \times BF$; likewise,

S B × B D : C A B D :: $2SA^3 : AB \times BF$.

Let $AS^2 \times SG = SB^3$ ∴ $AS^2 : SB^2 :: SB : SG$;

Because $BD : AC :: AS^3 : SB^3$ ($AS^2 \times SG$) :: A S : S G;

Ther. A C × A S = S G × B D;

Hence S B × B D : S A × A C :: S B × B D : S G × B D :: S B : S G :: $AS^2 : SB^2$. Ther. $AS^2 - SB^2 : SB^2$ :: S B × B D − S A × A C : S A × A C.

That is, F B × A B : SB^2 :: S B × B D − S A × A C : S A × A C;

But (Prop. XVII.) S B × B D − S A × A C = 2 × C A B D,

Ther. F B × A B : SB^2 :: 2 × C A B D : S A × A C,

Or $FB \times AB : 2SB^2 :: 2CABD : 2SA \times AC :: CABD : SA \times AC$.

Again, because $SB \times BD : SA \times AC :: AS^2 : SB^2 :: 2AS^2 : 2SB^2$;

And - - $S^2A \times AC : CABD :: 2SB^2 : FB \times AB$,

Therefore $SB \times BD : CABD :: 2SA^2 : FB \times AB$. Q.E.D.

Prop. XIX. Fig. 19.

The law of centripetal force being given, and the quadrature of curves being granted, it is required to find the trajectory in which a body will move, that sets out from a given place, with a given velocity, in the direction of a given right line.

Suppose AEFG to be the trajectory: with the center C, and distance CA, describe the circle APQ; draw AY, CZ, perpendicular to CA; with the asymptotes CA, CZ, let there be an hyperbola described passing through the point Y; let SRT be a curve, such, that drawing any line CFP, meeting the trajectory in F, and the circle in P, and with the center C, and distance CF, a circle be described meeting CA in K, and KR be drawn perpendicular to CA, meeting the curve in R, the space SAKR will be double the sector ACP. Let fall CB perpendicular to AB, the given right line in which the body is projected; let AE, FG, be indefinitely little parts of the trajectory described in equal times; with the center C, and distances CE, CF, CG, describe the circles EH, FK, GL; and let the circle GL meet CF in M; let DA be the space through which the body would fall, so as to acquire, at the point A, a velocity equal to that with which the body is projected in the direction AB; let CX be to CK as the velocity that would be acquired in falling from D to K, to the velocity that would be acquired in falling from D to A; draw La parallel to KR, meeting the curve in a; join CG, and let it meet the circle APQ in Q; and let KR meet the hyperbola in V.

Because the body sets out at A, to describe the trajectory with the velocity acquired in falling from D to A, and CF is equal to

CK, the velocity at F will be equal (Prop. XIII.) to the velocity acquired in falling from D to K;

Ther. (hypoth.) vel. at A : vel. at F ;: CK : CX :: AE : FG;

But because AE, FG, are described in equal times, the triangles CAE, CFG are equal; ther. $AE \times CB = CF \times GM = CK \times GM$. $\therefore$ CB : CK :: GM : AE :: CX : FG,

Ther. $CB^2 : CX^2 :: GM^2 : FG^2$. $\therefore$ $CX^2 - CB^2 : CB^2 :: FG^2 - GM^2 : GM^2 :: FM^2 : GM^2$.

Again, because from the nature of the curve, SRT, the space SALa = 2 × sector CAQ, and SAKR = 2 × sector CAP, ther. RKLa = 2 × sector CPQ; and because KL is indefinitely little, $KR \times KL = CP \times PQ$,

Ther. KL : PQ :: CP : KR : CA : KR;

But PQ : GM :: CP : CM = CL = CK :: CA : CK :: KV : AY = CA;

Ther. KL : GM :: KV : KR :: FM : GM;

Hence, $KV^2 : KR^2 :: FM^2 : GM^2 :: CX^2 - CB^2 : CB^2$; ther. KR is given.

From whence may be deduced the following

Construction.

Draw AY, CZ, perpendicular to CA, and let AY be equal to CA; with the asymptotes CA, CZ, let there be an hyperbola described passing through the point Y; and let the body set out from A in the direction of the right line AB, with the velocity acquired in falling from D to A; draw CB perpendicular to AB; and let K be any point in AC; let CX be to CK, as the velocity that would be acquired in falling from D to K, to the velocity that would be acquired in falling from D to A; let SRT be a curve, such, that drawing KV perpendicular to AC, meeting the hyperbola in V, and the curve SRT in R, $KV^2 : KR^2 :: CX^2 - CB^2 : CB^2$; let AY meet the curve SRT in S;—with the center C, and distance CA, describe the circle APQ; draw CP, such, that the area SAKR may be double the sector CAP; and with the center C, and distance CK, describe a circle meeting CP in F : F will be a point in the trajectory. Q. E. I.

Prop. XX. Fig. 20.

Suppose a body, acted upon by a centripetal force tending to the center S, to descend in the right line A S; let there be a curve C D E, such, that if from any two points A, B, in A S, there be drawn A C, B C, perpendicular to A S, the centrip. force at A : centrip. force at B : : A C : B E. Suppose the body at the point B to be projected in the direction B F, perpendicular to S B, with the velocity that would be acquired in falling from A to B; and let the centrifugal force at the point B, arising from the projectile force, be greater than the centripetal force at B : It is evident that the body will recede from the center S : In S A take S G, equal to the greatest distance the body will recede from the center : The sum of the centrifugal forces that have acted upon the body while it receded from the center, through the space B G, will be equal to the sum of the centripetal forces that have acted upon the body in the same time.

Let $SB \times BF = 2 \times CABE$; in B E take B H, so that $BH^2 = CABE$; draw S K perpendicular to S B; with the asymptotes S B, S K, describe an hyperbola to pass through the point H; let B L M be the trajectory described by the body; suppose the body at the point L in the trajectory; and let L M, B O, be described in indefinitely little equal parts of time; join S L, S M, S O; with the center S, and distance S L, describe the arc L G meeting S A in G; and with the center S, and distance S M, describe the arc M N meeting S L in N; draw G D perpendicular to S A, meeting the curve C D E in D, and the hyperbola in P.

Because B O, L M, are described in equal times, and the velocity in a circle equable,

L M : B O : : vel. at L : vel. at B : : vel. at G : vel. at B : : $\sqrt{CAGD} : \sqrt{CABE}$ (Hypoth. and Prop. III.);

Ther. $LM^2 : BO^2 :: CAGD : CABE :: CAGD : BH^2$ (hypoth.); and because the triangles S L M, S B O, are equal, $BO^2 : MN^2 :: SL^2 (SG^2) : SB^2 :: BH^2 : GP^2$ (propor. of hyperbola); ther. $LM^2 : MN^2 :: CAGD : GP^2$. Hence it is

evident, that when $LM = MN$, $CAGD = GP^2$; and therefore $SL = SG$, will be the greatest distance the body will recede from the center S.

Let there be a curve FQ, so related to SA, that if from any two points B, G, there be drawn BF, GQ, perpendicular to SA, the $SG^2 : SB^2 :: BF : GQ$; produce GS to R, and let $SR = SG$.

But (Prop. XI.) centrif. force at B : centrip. force at B :: $2 \times CABE : SB \times BE :: SB \times BF : SB \times BE :: BF : BE$ (because $SB \times BF = 2 \times CABE$); therefore, because BE represents the centripetal force at B, BF will represent the centrifugal force at B; but (Prop. VIII.) the centrifugal forces are reciprocally as the cubes of the distances from the center: therefore it is evident, that while the body receded from the center through the space BG;

The sum centrif. forces : sum centrip. forces :: FBGQ : EBGD;

And because $BH^2 = CABE$, $GP^2 = CADG$;

Ther. $CABE : CAGD :: BH^2 : GP^2 :: SG^2 : SB^2$ (hyperbola);

Ther. $CABE : EBGD :: SG^2 : SG^2 - SB^2 = BR \times BG$;

Ther. $SB \times BF : EBGD :: 2SG^2 : BR \times BG$; but (Prop. XVIII.) $SB \times BF : FBGQ : 2SG^2 : BR \times BG$;

Ther. $SB \times BF : FBGQ :: SB \times BF : EBGD$, $\therefore FBGQ = EBGD$;

Therefore the sum of the centrifugal forces that have acted upon the body while it receded from the center, through the space BG, will be equal to the sum of the centripetal forces that have acted upon the body in the same time.

Corollary.

From this it is evident, that the sum of the centrifugal forces that act upon a body while it revolves from apsis to apsis, is equal to the sum of the centripetal forces that act upon it in the same time.

Prop. XXI. Fig. 21.

Let there be an ellipse whose transverse axis is A B, and foci D, E; produce D A to F; and let D F be equal to A B; upon D F let a semicircle be described; draw D G, D H, to any two points in the ellipse; and draw D K, D L, perpendicular to G K, H L, tangents to the ellipse at G and H; with the center D, and distances D G, D H, describe the circles G M, H N; draw M O, N P, perpendicular to A D; and join D O, meeting N P in Q; then will D K : D L :: N P : N Q.

Join E G; and draw E R perpendicular to the tangent G K. Because D G + G E = A B = D F, and D G = D M; ther. D M + G E = D F ∴ G E = D F − D M = F M; ther. D M : F M :: D G : G E; but D M × M F = M O² ∴ D M² : M O² :: D M : M F :: D G : G E; because K R is a tangent at G, the angles D G K, E G R, will be equal; and the angles at K and R being right, the triangles D G K, E G R, are equiangular; ther. D K : E R :: D G : G E :: D K² : D K × E R :: D M² : M O², but (26 Ellip.) D K × E R = A D × D B; ther. D K² : A D × D B :: D M² : M O².

In the same way it is shewn that D L² : A D × D B :: D N² : N P²;

Ther. A D × D B : D L² :: N P² : D N²;

But D K² : A D × D B :: D M² : M O² :: D N² : N Q²;

Ther. D K² : D L² :: N P² : N Q²; or, D K : D L :: N P : N Q. Q. E. D.

Corollary. D K² : A D × D B :: D G : G E.

Prop. XXII. Fig. 22.

Let there be a semicircle whose diameter is A B; and let A D E be a curve, such, that if from any two points D, E, there be drawn D C, E F, perpendicular to A B, meeting the semicircle in G, H, and joining B G, meeting F H in K, E F : D C :: F H² : F K², the curve A D E, will be an hyperbola: and let D L, E M be tangents at D, E; E F × C L : D C × F M :: B C² : B F².

Bisect AB in C; produce EF, so that FO=DC; through O, draw PN parallel to AB, meeting AN, BP, perpendicular to AB, in N, P; and DC in R.

Because EF : DC :: FH² : FK²; and FH²=AF×FB, and FK²=FB² (because CG=CB) therefore, EF : DC (FO) :: AF×FB : FB² :: AF : FB;

Hence EF+FO : FO :: AF+FB : FB;

That is, EO : FO (AN) :: AB : FB :: NP : PO ∴ EO × PO=AN×NP;

Therefore the curve ADE is an hyperbola whose asymptotes are PB and PN.

Again, let DC, and the tangents DL, EM, meet PN in R, N, Q. Because a tangent drawn at any point in an hyperbola to meet the asymptotes is bisected in the point of contact; NR =RP, and QO=OP; and because EF : FM :: EO : OQ (OP) :: EF×LC : FM×LC :: EO² : OP×EO. Again, because LC : CD :: NR : RD :: PR : RD :: LC×FM : CD× FM :: PR×RD : RD² :: EO×OP : RD² (prop. of hyperbola); but EF×LC : FM×LC :: EO² : OP×EO; ther. EF ×LC : CD×FM :: EO² : RD²; because EO×OP=DR× RP, EO : DR :: RP : OP :: CB : BF ∴ EO² : DR² :: CB² : BF² :: EF×LC : CD×FM. Q. E. D.

Prop. XXIII. Fig. 22.

Suppose a body to descend or ascend in the line AB; let C, F, be any two points in AB; draw CG, FH, perpendicular to AB, meeting the semicircle described upon AB in G, H; join BG meeting FH in K. If the vel. at F : vel. at C :: FH : FK, then will the centrip. force at F : centrip. force at C :: BC² : BF².

Let ADE be a curve, such, that if from any two points D, E, there be drawn DC, EF, perpendicular to AB, meeting the semicircle in G, H, and joining BG, meeting FH in K; EF : CD :: HF² : FK²; and let DL and EM be tangents to the curve at D, E.

Because HF² : FK² :: EF : CD ∴ HF : FK :: $\sqrt{EF}$:

$\sqrt{CD}$; but, vel. at F : vel. at C :: HF : FK :: $\sqrt{EF}$: $\sqrt{CD}$; ther. (Prop. V.) centrip. force at F : centrip. force at C :: EF × CL : DC × FM :: BC^2 : BF^2 (Prop. XXII.) Q. E. D.

Prop. XXIV. Fig. 21.

If a body revolve in an ellipse, and be acted upon by a centripetal force tending to the focus of the ellipse, the centripetal force will be reciprocally as the square of the distance from the focus.

Let a body revolve in the ellipse whose transverse axis is AB, and be acted upon by a centripetal force tending to the focus D; produce DA, so that DF = AB; draw DG, DH, to any two points in the ellipse; draw DK, DL, perpendicular to the tangents at G and H; with the center D, and distances DG, DH, describe the circles GM, HN; draw MO, NP, perpendicular to AD, meeting the semicircle described upon DF in O, P; join DO meeting NP in Q.

Suppose a body to descend or ascend in the line AD, and at the point A to have acquired a velocity equal to the velocity which the body describing the ellipse has at the point A; the velocities of the bodies at G and M will be equal (Prop. XIII.); and likewise the velocities of the bodies at H and N will be equal: but (Prop. XII.) vel. at H : vel. at G :: DK : DL :: NP : NQ. (Prop. XXI.)

Ther. vel. at N : vel. at M :: NP : NQ;

Ther. (Prop. XXIII.) centrip. force at N : centrip. force at M :: DM^2 : DN^2 :: DG^2 : DH^2; because DG = DM, DH = DN, and GM, HN, circles; the centrip. force at G is equal to the centrip. force at M; and the centrip. force at H equal to the centrip. at N;

Ther. centrip. force at H : centrip. force at G :: DG^2 : DH^2 :: $\frac{1}{DH^2} : \frac{1}{DG^2}$.

Otherwise, Fig. 23.

Let S be the focus of the ellipse, AB the greater axis, C the center, CD half the less axis; suppose a body in indefinitely

little equal parts of time to describe AE, FG; join SE, SF, SG; draw EK perpendicular to AB, and GM perpendicular to SF; let FP be a tangent at F; draw GP parallel to SF; join CF meeting the ellipse in H; and let GN, drawn parallel to FP, meet SF, CF, in N, O; draw CQ parallel to FP meeting SF in V; complete the parallelogram FCQT; and draw FR perpendicular to CQ.

Because QV, GN are parallel, and the angles at M and R, right, the triangles FRV, GMN are similar; ther. MG : GN :: RF : FV=CA (ellip. 25. cor.) : but (ellip. 31. cor. 2.) RF×CQ=CA×CD; ther. RF : CA :: CD : CQ :: MG : GN; but GN, GO may be considered as equal, because FG is described in an indefinitely little part of time; ther. $MG^2 : GO^2$ :: $CD^2 : CQ^2$: but (ellip. 16.) $GO^2 : HO \times OF :: CQ^2 : CF^2$ ∴ $MG^2 : HO \times OF :: CD^2 : CF^2$; but HO×OF, may be considered equal to HF×FO; ther. $GM^2 : HF \times FO :: CD^2 : CF^2$; but HF×FO : HF×FN :: FO : FN :: FC : FV (CA) :: $CF^2 : CA \times FC$ ∴ $GM^2 : HF \times FN :: CD^2 : CA \times CF$;

But HF×FN : AB×FN :: HF : AB :: CF : CA :: FC×CA : CA^2;

Ther. $GM^2 : AB \times FN :: CD^2 : CA^2 :: EK^2 : AK \times KB$ (ellip. 3.)

But BK×KA, may be considered equal to BA×AK;

Ther. $GM^2 : AB \times FN :: EK^2 : BA \times AK$;

That is, $GM^2 : EK^2 :: AB \times FN : AB \times AK :: FN : AK$;

But because AE, FG, are described in equal times the triangle SAE=SFG; that is, SA×EK=SF×GM; ∴ $GM^2 : EK^2 :: SA^2 : SF^2$;

Ther. FN(GP) : AK :: SA^2 : SF^2; that is, GP : KA :: $\frac{1}{SF^2} : \frac{1}{SA^2}$.

Therefore the centripetal force is reciprocally as the square of the distance from the focus. Q. E. D.

Prop. XXV. Fig. 24.

If a body revolve in an ellipse, it is required to find the law of centripetal force tending to the center of the ellipse.

Let AB be the greater axis, C the center, and CD half the lesser axis; and suppose a body in indefinitely little equal parts of time to describe AE, FG; join CF, CE, CG; draw EK perpendicular to AB, and GM perpendicular to CF; let FP be a tangent at F; draw GP parallel to CF, and GN, CQ parallel to FP; complete the parallelogram FCQT; draw FR perpendicular to CQ, and let CF meet the ellipse in H.

Because the parts AE, FG, are described in equal times, $CA \times EK = CF \times GM$, $\therefore EK^2 : GM^2 :: CF^2 : CA^2$; because GN is parallel to CQ, and the angles at M and R are right, the triangles GNM, FCR, are equiangular;

Ther. $GM^2 : GN^2 :: FR^2 : FC^2 \therefore EK^2 : GN^2 :: FR^2 : CA^2 :: CD^2 : CQ^2$ (ellip. 31. cor. 2.); but (ellip 16.) $GN^2 : HN \times NF :: CQ^2 : CF^2$;

Ther. $EK^2 : HN \times NF :: CD^2 : CF^2 \therefore HN \times NF : EK^2 :: CF^2 : CD^2$:

But (ellip. 3.) $EK^2 : BK \times KA :: CD^2 : CA^2 \therefore HN \times NF : BK \times KA :: CF^2 : CA^2$;

But because AE, FG, are indefinitely little, we may consider $HN \times NF = HF \times FN$, and $BK \times KA = BA \times AK$;

Ther. $HF \times FN : BA \times AK :: CF^2 : CA^2 :: CF \times FN : CA \times AK$;

Ther. $FN(GP) : AK :: CF : CA$;

But GP : AK :: centrip. force at F : centrip. force at A :: CF : CA.

Therefore the centripetal force tending to the center of the ellipse is as the distance from the center of the ellipse. Q.E.I.

Prop. XXVI. Fig. 25.

Let there be an ellipse whose transverse axis is AB; in which let F be a given point; suppose a body acted upon by a centripetal force, tending to the point F, to describe the ellipse: it is required to determine the law of centripetal force tending to the point F.

Let C be the center of the ellipse, and E the focus; and let C D be half the less axis; let K L, A G, be described in indefinitely little equal parts of time; join F K, C K, E K, and F L; draw L P perpendicular to F K; draw L M parallel to F K, meeting the tangent K M in M; draw L N parallel to K M, meeting F K, C K, in N, O; draw C Q parallel to K M, meeting F K, E K, in Q, X; and draw F Y parallel to C Q or K M; let C K, C Q, meet the ellipse in T, R; and draw K S perpendicular to C R, meeting F Y in Z; draw G H perpendicular to A B; join F G; and let C A × K Y = A F × F V.

Because L N, F Z, are parallel, and the angles at P and Z right; the triangles L P N, F K Z, are similar; ther. N L : L P :: F K : K Z . ∴ N L × K Z = L P × F K; and because K L, A G, are described in equal times, ther. L P × F K = G H × F A = N L × K Z . ∴ L N : G H :: A F : K Z :: A F × C R : K Z × C R; but K Z : K Y :: K S : K X (C A) :: C D : C R;

Ther. K Z × C R = C D × K Y . ∴ L N : G H :: A F × C R : C D × K Y;

Ther. L N² : G H² :: A F² × C R² : C D² × K Y²;

But G H² : B H × H A :: C D² : C A² :: C D² × K Y² : C A² × K Y²;

Ther. L N² : B H × H A :: A F² × C R² : C A² × K Y² = A F² × F V² (hypoth.);

Ther. L N² : B H × H A :: C R² : F V²;

But because K L is indefinitely little, we may consider L N = L O, ther. B H × H A : L O² :: F V² : C R²;

But L O² : T O × O K :: C R² : C K² . ∴ B H × H A : T O × O K :: F V² : C K²;

But because A G, K L, are indefinitely little, we may consider B H × H A = B A × A H, and T O × O K = T K × K O;

Ther. B A × A H : T K × K O :: F V² : C K²;

But T K × K O : T K × K N :: K O : K N :: K C : K Q :: K C² : K Q × K C, ther. B A × A H : T K × K N :: F V² : C K × K Q;

But T K × K N : A B × K N :: C K : C A :: C K × K Q : C A × K Q;

Ther. B A × A H : A B × K N :: F V² : C A × K Q;

Ther. $AH : KN :: FV^2 : AC \times KQ$.

Because $CA \times KY = AF \times FV$.·. $FV^2 : CA^2 :: KY^2 : AF^2 :: KY^2 \times FK : AF^2 \times FK$; but $CA^2 : CA \times KQ :: AC : KQ :: KX : KQ :: KY : KF :: KY^3 : KY^2 \times KF$; ther. $FV^2 : CA \times KQ :: KY^3 : AF^2 \times FK$; ther. $AH : KN :: KY^3 : AF^2 \times FK$; but because GH is parallel to the tangent at A, and LN parallel to the tangent at K; and AG, LK, are indefinitely little, and described in equal times.

The centrip. force at A : centrip. force at K :: $AH : KN :: KY^3 : AF^2 \times KF$. Q.E.D.

Coaollary. This proposition comprehends the two former ones; for if the point F coincide with the focus E, the point Y will coincide with E; and $KY^3 = KE^3$; likewise $AF^2 \times KF = AE^2 \times KE$.·. $AH : KN :: KE^3 : AE^2 \times KE :: KE^2 : AE^2$; which agrees with Prop. XXIV.

When F coincides with C, $KY = KX = CA$;

$AF = AC$, and $FK = CK$; therefore

$AH : KN :: CA^3 : CA^2 \times CK :: CA : CK$; which agrees with Prop. XXV.

Prop. XXVII. Fig. 26.

If a body describe an hyperbola, it is required to find the law of centripetal force tending to the focus.

Let S be the focus of the hyperbola, AB the transverse axis, C the center, and CD half the second axis; let AE, FG, be described in indefinitely little equal parts of time; join SE, SF, SG; draw EK perpendicular to SA, and GM perpendicular to SF; let FP be a tangent at F; draw GP parallel to SF; join CF, which produce, so that $CF = CH$; let GN, drawn parallel to FP, meet SF, CF, in N, O; let CQ, the semi-diameter conjugate to CF, meet SF in V; complete the parallelogram FCQT; and draw FR perpendicular to CQ.

Because QV, GN, are parallel, and the angles at R and M right; the triangles FRV, GMN, will be similar; ther. $GM : GN :: FR : FV = CA$; but the parallelogram $FCQT = CQ \times FR = AC \times CD$, .·. $FR : CA :: CD : CQ :: GM : GN$; but

because F G is described in an indefinitely little part of time, we may consider G N = G O; ther. $GM^2 : GO^2 :: CD^2 : CQ^2$; but $GO^2 : HO \times OF :: CQ^2 : CF^2$; ther. $GM^2 : HO \times OF :: CD^2 : CF^2$; we may consider $HO \times OF = HF \times OF$; therefore, $GM^2 : HF \times OF :: CD^2 : CF^2$; but $HF \times FO : HF \times FN :: FO : FN :: CF : FV(CA) :: CF^2 : FC \times CA$; ther. $GM^2 : HF \times FN :: CD^2 : FC \times CA$; but $HF \times FN : AB \times FN :: HF : AB :: CF : CA :: CF \times CA : CA^2$; ther. $GM^2 : AB \times FN :: CD^2 : CA^2 :: EK^2 : BK \times KA$; but we may consider $BK \times KA = BA \times AK$; therefore, $GM^2 : AB \times FN :: EK^2 : BA \times AK$; that is, $GM^2 : EK^2 :: FN : AK$; but because A E, F G, are described in equal times, $SA \times EK = SF \times GM$; therefore $GM^2 : EK^2 :: SA^2 : SF^2 :: FN : AK :: GP : AK$;

Hence the centrip. force at F : centrip. force at A :: G P : $AK :: SA^2 : SF^2 :: \frac{1}{SF^2} : \frac{1}{SA^2}$; therefore the centripetal force is reciprocally as the square of the distance from the focus S. Q. E. I.

Prop. XXVIII. Fig. 27.

If a body describe a parabola, it is required to find the law of the centripetal force tending to the focus of the parabola.

Let S be the focus of the parabola, A B the axis, and A the vertex; let A E, F G, be described in indefinitely little equal parts of time; join S E, S F, S G; draw E K perpendicular to S A, and G M perpendicular to S F; let F P, a tangent at F, meet S A in R; draw F C parallel to A B, and G P parallel to S F, and G N parallel to F P, meeting S F, F C, in N, O; and draw S Q perpendicular to F R.

Because G N, F Q, are parallel, and the angles at M and Q right; the triangles G N M, S F Q, are similar; ther. N G : G M : : S F : S Q; because F G is described in an indefinitely little part of time, we may consider G N = G O; ther. $OG^2 : GM^2 :: SF^2 : SQ^2 :: SF^2 : SA \times SF :: SF : SA$ (Prop. of Parabola.)

Again, because $GO^2 = FO \times$ (Param. of C F) $= FO \times 4SF$;

ther. $4 \times SF \times FO : GM^2 :: SF : SA :: 4SF \times FO : 4SA \times FO$; ther. $GM^2 = 4SA \times FO$; but $EK^2 = 4SA \times AK$; ther. $GM^2 : EK^2 :: FO : AK$; but because NO is parallel to FR, and FO parallel to SR, the triangles FNO, FSR, are similar; ther. $NF : FO :: FS : SR$; but $SF = SR$, ther. $FN = FO$ $\therefore$ $GM^2 : EK^2 :: FN : AK$; because AE, FG, are described in equal times, therefore $SA \times EK = SF \times GM$ $\therefore$ $GM^2 : EK^2 :: SA^2 : SF^2 :: FN : AK$; but because $FN = GP$; $SA^2 : SF^2 :: GP : AK :: \frac{1}{SF^2} : \frac{1}{SA^2}$; therefore the centripetal force is reciprocally as the square of the distance from the focus S. Q.E.I.

ADDENDA.

COROLLARIES TO PROP. XLIV. OF THE ELLIPSE.

Corollary 1.

From hence it appears, that the conjugate diameter K W produced, is the *locus* of the centers of curvature of every degree that is perpendicular to, and bisected by, the meridian I K H W.

Corollary 2.

When the angle P X H=45°, then t=1, and the secant = $\sqrt{2}$; we then have $v\,s = \frac{2 \cdot C\,H^2 \cdot C\,K}{C\,H^2 + C\,K^2}$, $v\,P = \frac{2\sqrt{2} \cdot CH^2 \cdot CK^2}{(C\,H^2 + C\,K^2)^{\frac{3}{2}}}$, and the radius of curvature of a degree perpendicular to the meridian in lat. 45° $= \frac{\sqrt{2} \cdot C\,H^2}{(C\,H^2 + C\,K^2)^{\frac{1}{2}}}$.

Corollary 3.

When P coincides with K, the angle P X H = 90°, and both the tangent and secant become infinite in length.

Then $v\,s = \frac{C\,H^2 \cdot C\,K \cdot (inf.)^2}{\sqrt{(inf.)^2 \cdot C\,K^2} \times \sqrt{(inf.)^2 \cdot C\,H^2}} = C\,H.$

$v\,P = \frac{C\,H^2 \cdot C\,K^2 \cdot (inf.)^3}{(inf.)^2 \cdot C\,H^2 \cdot \sqrt{(inf.)^2 \cdot CK^2}} = C\,K$, and half the latus-rectum of $P\,Z = \frac{C\,H^2 \cdot inf.}{\sqrt{(inf.)^2 \cdot C\,K^2}} = \frac{C\,H^2}{C\,K}$.

COROLLARY 4.

When P coincides with H, the angle P X H is nothing, its secant unity, and t=0; then

$$v\,s = \frac{C\,H^2 \cdot C\,K}{\sqrt{C\,H^2} \cdot \sqrt{C\,K^2}} = C\,H, \quad v\,P = \frac{C\,H^2 \cdot C\,K^2}{C\,K^2 \cdot \sqrt{C\,H^2}} = C\,H,$$

and half the latus-rectum of $P\,Z = \frac{C\,H^2}{\sqrt{C\,H^2}} = C\,H$. In this case the section P t Z s, evidently becomes a circle.

COROLLARY 5.

When the section I K H W is a circle, or C H = C K;

Then $v\,s = \frac{(1+t^2)\,C\,H^3}{\sqrt{(1+t^2)\,C\,H^2} \times \sqrt{(1+t^2)\,C\,H^2}} = C\,H,$

$$v\,P = \frac{(1+t^2)\,(1+t^2)^{\frac{1}{2}} \cdot C\,H^4}{(1+t^2)\,C\,H^2\,\sqrt{(1+t^2)\,C\,H^2}} = C\,H,$$ and half the

latus-rectum of $P\,Z = \frac{(1+t^2)^{\frac{1}{2}} \cdot C\,H^2}{\sqrt{(1+t^2)\,C\,H^2}} = C\,H.$

PROPOSITION A.

The length of a degree perpendicular to a meridian, is greater than a corresponding meridional degree.

By Prop. XLIV. of the Ellipse, the radius of curvature of a degree perpendicular to the meridian in any given latitude is

$$\frac{C\,H^2}{c(C\,H^2+t^2 \cdot C\,K^2)^{\frac{1}{2}}} = R,$$

And by Cor. 1. page 61, the radius of curvature of a meridional degree in any given latitude is $\frac{C\,H^2 \cdot C\,K^2}{\left\{c(CH^2+t^2 \cdot CK^2)^{\frac{1}{2}}\right\}^3} = P\,\odot,$

from whence $R : P\,\odot :: 1 : \frac{C\,K^2}{c^2(C\,H^2+t^2 \cdot C\,K^2)} :: \frac{C\,H^2}{C\,K^2} + t^2 : \frac{1}{c^2} :: \frac{C\,H^2}{C\,K^2} + t^2 : 1 + t^2.$

Therefore, in any given latitude, the length of a degree perpendicular to the meridian is greater than the length of a corresponding meridional degree; the lengths of the degrees being as the radii of curvature and $\frac{C H^2}{C K^2}$ greater than unity.

COROLLARY 1.

Since $D' : D :: \frac{C H^2}{C K^2} + t^2 : 1 + t^2$, we get $\frac{C H^2}{C K^2} = \frac{D + (D' - D) t^2}{D}$.

Hence the ratio of the axes are known.

D′ representing the length of a degree perpendicular to the meridian, and D the length of a corresponding meridional degree.

COROLLARY 2.

A the equator, $t = 0$, and $D' : D :: \frac{C H^2}{C K^2} : 1 :: C H^2 : C K^2$.

COROLLARY 3.

In lat. 45°, $t = 1$, and $D' : D :: \frac{C H^2}{C K^2} + 1 : 2$.

COROLLARY 4.

At the poles, $t =$ tang. 90°, and therefore infinitely long; then $D' : D :: \frac{C H^2}{C K^2} + (\text{inf.})^2 : 1 + (\text{inf.})^2 :: (\text{inf.})^2 : (\text{inf.})^2 \therefore D' = D$.

PROPOSITION B.

Having the meridional degree, and also the degree perpendicular to the meridian in a given latitude, to find the earth's axes, supposing it an ellipsoid. See Prop. XVI. page 66.

By Cor. 1. Prop. A. we have - - - $\frac{C K^2}{C H} = \frac{C H}{1 + m t^2}$ - - (a).

And from Prop. IV. page 57 - - - $\frac{C K^2}{C H} = v D \cos.^3 P f T$;

From whence $CH = vD(1+mt^2)\cos.^3 PfT$ (fig. page 65, $y = 57{\cdot}2957795$, and $\frac{D'-D}{D} = m$). From equation (a) $CH : \sqrt{CH^2 - CK^2} :: \sqrt{1+mt^2} : t\sqrt{m}$, and per trigonometry $\sqrt{1+mt^2} : t\sqrt{m} :: s : \frac{st\sqrt{m}}{\sqrt{1+mt^2}} = \sin. PfT$, therefore $\left(\frac{1+ms^2}{1+mt^2}\right)^{\frac{3}{2}} = \cos.^3 PfT$ ($s = \sin. t = \tang. EfH$). By substitution, $CH^2 = \frac{v^2D^2(1+ms^2)^3}{1+mt^2}$, and $CK^2 = \frac{v^2D^2(1+ms^2)^3}{(1+mt^2)^2}$.

FINIS.

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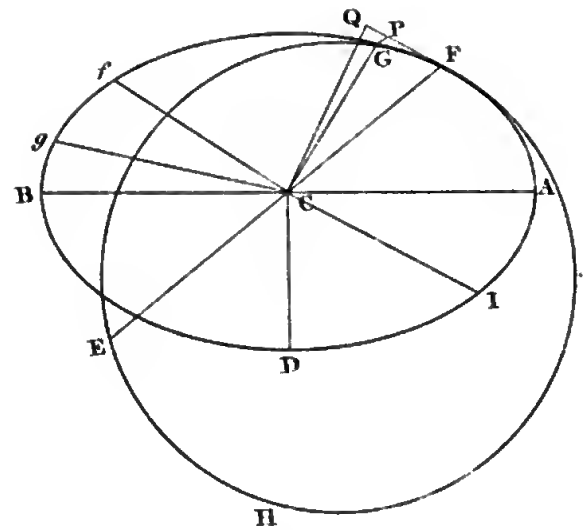
ERRATA.

Page.	Line.	Error.	Correction.
9	7	Ff	F, f
10	4	$GH : CH$	$CH :: CH$
14	9	$\frac{QF^2}{QE^2}$	$\frac{QF^2}{Qe^2}$
17	- - -	Lemma 1.	Lemma
26	4	$Pd \times CG = CW^2$	$PD \times CG = CW^2$
29	4	$CR\,CK$	CK
32	9	$CI^2, EI^2 - CG^2 + CI^2 - CE^2$	$CI^2 - CG^2 + CI^2 - CE^2$
36	15	$\sin.^2 EDP = PD^2$	$\sin.^2 EDP \times PD^2$
38	15	$CL : CW : gm : gr$	$CL : CW :: gm : gr$
41	last	$Fr -$	$Fr =$
43	19	$CF \times \cos. RFH$	$CF \times \cos. RfH$
46	7	Common section PZ	normal PX
46	Fig.	k	h, and join PC
48	2		dele 1 in denominator
48	4	vortex	vertex
49	4	TPf	TPF
49	5	cor. 8	cor. 1
49	Fig.		P is omitted
50	7	Lemma 2.	Lemma
51	5	ther.	then
59	last	Add to the last line	(see Fig. to Prop. IV.)
65	last	$(a^2 - v^2 S^2)^3$	$(a^2 - v^2 S^2)^{\frac{1}{2}}$
90	14	Prop. XXXVII.	Prop. XXXVII. ellipse
99	23		dele $A \cdot C$
106	21	vel. at F : vel. at F	vel. at F : vel. at f
110	1	$a\,b^2 :: d\,n^2$	$a\,b^2 \times d\,n^2$
113	9	$\frac{GL^2}{SL} : \frac{gl^2}{SL}$	$\frac{GL^2}{SL} \cdot \frac{gl^2}{SI}$
123	6	FG	F, G

Fig. 2.



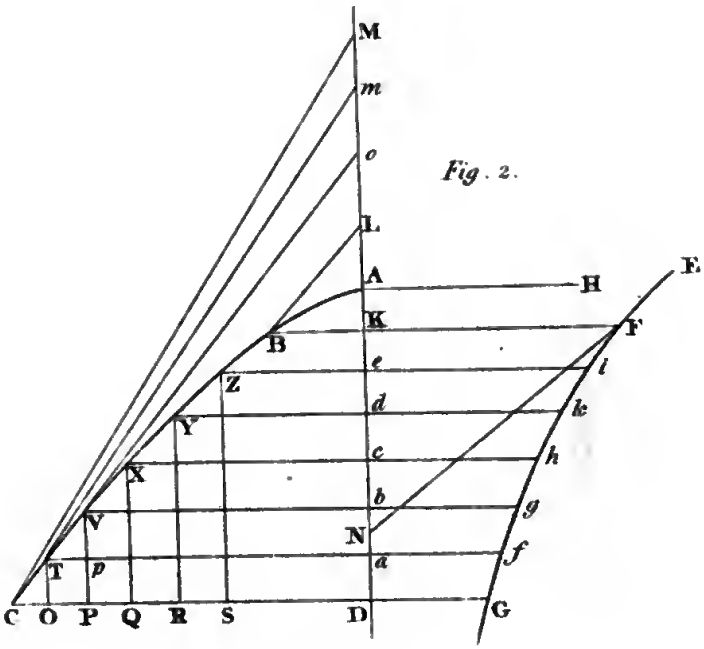

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Fig. 2.

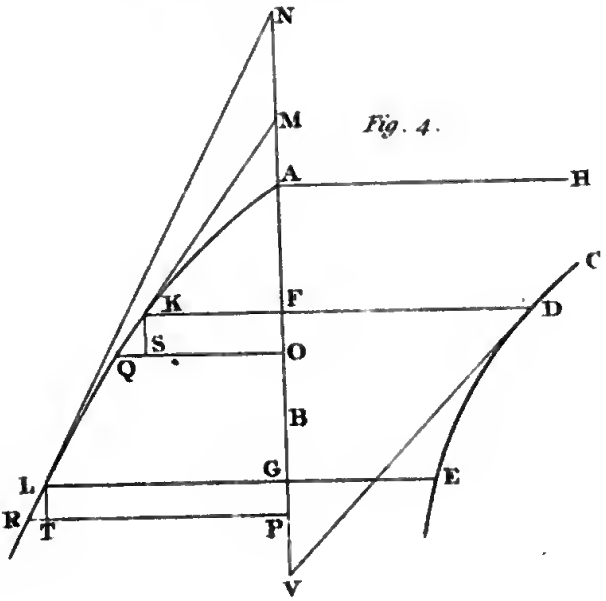
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Fig. 4.

Fig. 7.

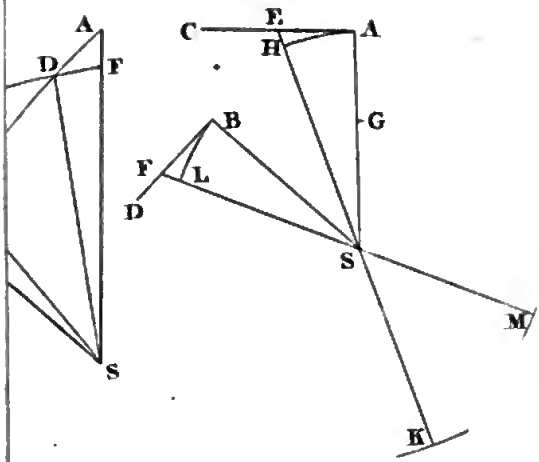
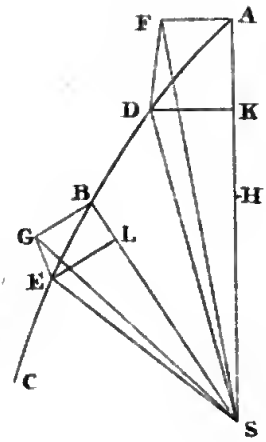
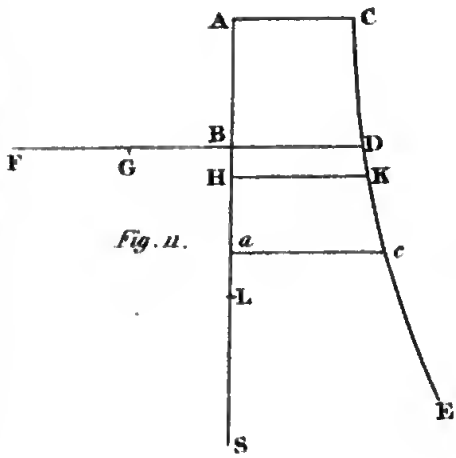


Fig. 8.





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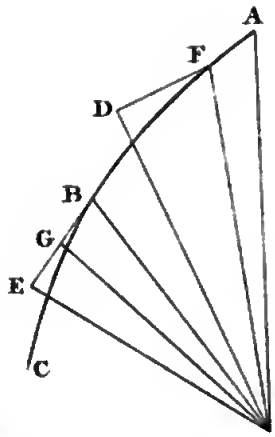
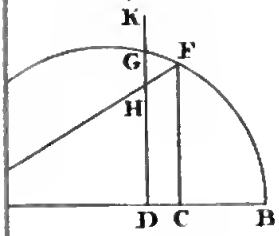


Fig. 12.

Fig. 15.



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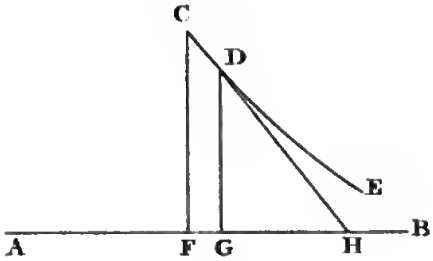

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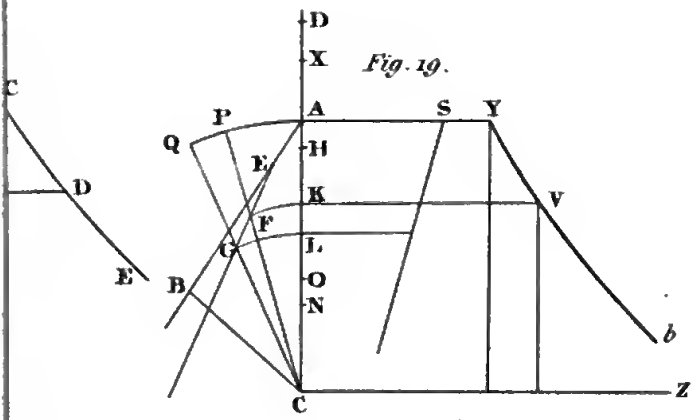
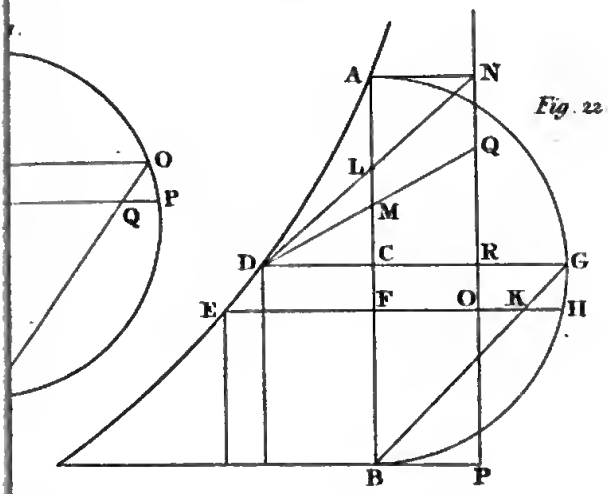


Fig. 22



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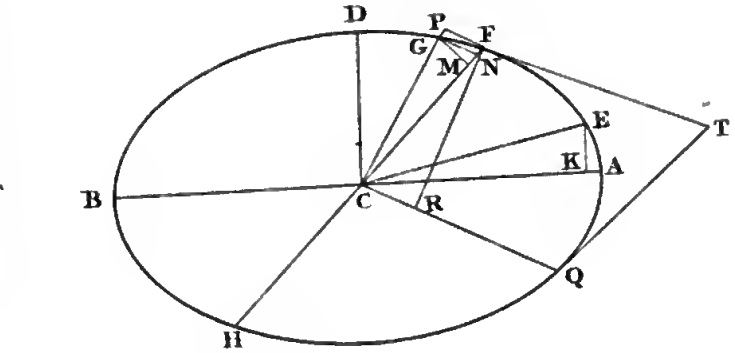

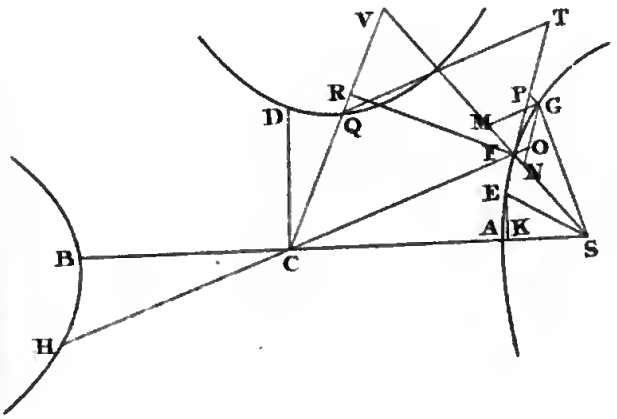

Fig. 26.

BIBLIOTECA